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I.

BINARY FAMILIES IN A TRIPLY CONNECTED REGION, WITH ESPECIAL REFERENCE TO HYPERGEOMETRIC FAMILIES.

BY

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Presented by Maxime Bôcher, May 13, 1903.

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BINARY FAMILIES IN A TRIPLY CONNECTED REGION WITH ESPECIAL REFERENCE TO HYPERGEOMETRIC FAMILIES.[1]

INTRODUCTION.

It was Riemann who, in his celebrated paper on hypergeometric functions,[2] first emphasized the importance of the group of substitutions which such a function undergoes when continued along all possible paths which do not pass through a singular point. Again, in another paper,[3] he discusses from this same point of view families composed of the solutions of homogeneous linear differential equations with algebraic coefficients. In the present memoir his methods are applied, and extended, where necessary, to the study of the properties of *binary families* (a term to be explained presently), particularly those whose members are analytic in a triply connected region. In Part I the properties of the most general families of this sort are discussed, while Part II is devoted to the special case of binary families whose only singularities are three *regular singular points,* i.e. *hypergeometric families.*

Since the solutions of a homogeneous linear differential equation of the second order with single valued coefficients constitute a binary family, it would, from one point of view, have been most natural to make this discussion a chapter in the theory of such equations; but we have preferred to follow the methods of Riemann. It is important, however, to note that all our work may be stated in terms of the differential equation; in particular the classification of binary families in a triply connected region given in the last half of Part I may be regarded as a classification of the sub-groups of differential equations of the above kind, obtained from the general group of the equation by so introducing cross-cuts in the region of definition of the coefficients as to make it triply connected.

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[1] This paper was accepted in June, 1903, by the Faculty of Arts and Sciences of Harvard University in satisfaction of the requirement of a thesis for the degree of Doctor of Philosophy.

[2] Beiträge zur Theorie der durch die Gauss'sche Reihe $F(\alpha, \beta, \gamma, x)$ darstellbaren Functionen. (1857). Werke, p. 67.

[3] Zwei allgemeine Sätze über lineare Differentialgleichungen mit algebraischen Coefficienten. (1857). Werke, p. 379.

The methods employed in this classification are believed to be entirely new, though forming a natural development of Riemann's work. Ritter's classification[1] of hypergeometric families without apparently singular points, which in Part II, D, we give as a corollary of our more general results, and which he obtained by a method totally different from ours, also furnished valuable suggestions. The case he treated is the most general for which such a classification has hitherto been attempted.

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[1] In § 2 of his posthumous memoir: Ueber die hypergeometrische Function mit einem Nebenpunkt. (1896). Math. Ann., vol. 48.

PART I.

THE GENERAL CASE.

WHILE this memoir refers primarily to families belonging to a triply connected region, there are many ideas concerned which apply equally well to regions of higher connectivity. We will develop these more general ideas in a subdivision A, following this with a subdivision B, devoted to the more special properties true for the triply connected region alone.

A. BINARY FAMILIES IN AN n-TUPLY CONNECTED REGION.

§ 1. THE REGIONS T_n AND S.

Let $a, b, c, \ldots, l$ be n simply connected perfect[1] regions on the complex sphere, no two of which have a point in common. The remainder of the sphere constitutes the n-tuply connected region T_n with *holes* $a, b, c, \ldots, l$. These holes may be continuous open line-segments, or even points, as well as two-dimensional regions. By $n-1$ cross-cuts joining a to b, b to c, etc., we can convert T_n into a simply connected region which we shall refer to as the region S.

§ 2. DEFINITION OF A BINARY FAMILY.

In the region T_n we are to consider a family of functions which we shall call a *binary family*, defined as follows: —

(1) Every member is single valued and analytic throughout S, and can be continued over the cross-cuts, the branches thus obtained being again members of the family; i. e. we have to do with a family of function-branches analytic in T_n.

(2) There are two linearly independent branches, y_1, y_2, such that every member can be expressed in the form $c_1 y_1 + c_2 y_2$, where c_1 and c_2 are constants; and, conversely, if we give to c_1 and c_2 any values whatever, the function $c_1 y_1 + c_2 y_2$ belongs to the family.

We shall call the pair (y_1, y_2) a *basis* of the family. Any pair of linearly independent branches is readily seen to constitute a basis.

---

[1] By a perfect region is understood a region which includes all of its limiting points.

§ 3. THE GROUP OF A BINARY FAMILY.

Starting from any given point of S, let us continue simultaneously a chosen basis (y_1, y_2) along a closed path which may, if we please, intersect one or more of the cross-cuts, but must lie within T_n. After making this circuit, y_1 and y_2 have become branches $\bar{y}_1$ and $\bar{y}_2$, linearly independent, and still members of the family. Definition (2) of the preceding section gives us the equations

$$\begin{aligned} \bar{y}_1 &= \alpha y_1 + \beta y_2 \\ \bar{y}_2 &= \gamma y_1 + \delta y_2, \end{aligned}$$

where $\alpha, \beta, \gamma, \delta$ are constants, and $\alpha\delta - \beta\gamma \neq 0$. *The basis* (y_1, y_2) *has undergone the linear substitution*

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix};$$

symbolically

$$(\bar{y}_1, \bar{y}_2) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} (y_1, y_2) = A\,(y_1, y_2).$$

If we had continued our basis along some other closed path in T_n, (y_1, y_2) would have undergone, in general, another linear substitution

$$B = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}.$$

If we continue (y_1, y_2) along the first path, and then along the second, it undergoes the linear substitution

$$A\,B = \begin{pmatrix} \alpha\alpha' + \beta\gamma' & \alpha\beta' + \beta\delta' \\ \gamma\alpha' + \delta\gamma' & \gamma\beta' + \delta\delta' \end{pmatrix}.$$

For a closer treatment of this subject, one may consult Klein's *Hypergeometrische Functionen.*[1] The points of chief importance are these: Corresponding to the fact that all closed paths in T_n may be regarded as obtained by combination and repetition of n closed *fundamental paths,* namely single circuits about each hole alone, we have the theorem that all the substitutions which a basis (y_1, y_2) can undergo are combinations and repetitions of n *generating substitutions,* each of which corresponds to a single circuit about one, and only one, of the holes $a, b, c, \ldots, l$, respectively.

At this point, however, we should note a matter to which Riemann alone of the writers on this subject seems to have given especial attention. This is the fact that in the above definition of a fundamental path, and hence of a generating substitution, we still need to specify how such a path is to be taken relatively to the cross-cuts which change T_n into S; this is a matter of convention, but unless we add such a specification, inconsistencies will arise in the computations which follow. The convention we here adopt is practically Riemann's.[1] — Given a point P from which a fundamental path about hole i is to start and to which it is to return. We divide S into two regions by joining l to a by a cross-cut such that that part of S in which P is lies to the left of a point which traces out its boundary so that it meets the holes in the order $l, \ldots, c, b, a$. A fundamental path about i is to have the hole i, but no other hole, interior to it, and is to be deformable into a path which crosses the cross-cuts at only two points. A generating substitution corresponds to such a path traversed positively, its inverse to that path traversed negatively.

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[1] Lithographed lectures 1893–94, pp. 98–101. We shall have occasion to refer to this volume several times, and, as this is the only work of Klein's here cited, we shall hereafter use merely the author's name in referring to it.

With this convention we see that a path formed of n fundamental paths about $l, \ldots, c, b, a$, successively, each taken positively, is equivalent to a path enclosing no hole. If the n generating substitutions for a given basis are $S_a, S_b, S_c, \ldots, S_l$, this gives the relation

$$S_l \ldots S_c S_b S_a = 1, \tag{1}$$

where 1 stands for the identical substitution.

A different convention would have changed the order of the product in (1).

The totality of substitutions which a basis undergoes when continued along all possible closed paths in $\mathrm{T_n}$ *constitutes a group whose* n *generating substitutions correspond to single circuits about each of the holes.*

This group, which we call the group of the binary family, is discontinuous, and, in general, infinite. If we had used instead of (y_1, y_2) any other basis (y_1', y_2'), we should have obtained a new group simply isomorphic with the old, and therefore, as an abstract group, indistinguishable from it. In so far we are justified in speaking of the group of the *family* as we have done; if there is need of a more specific expression, we shall speak of the *group corresponding to a basis.*

§ 4. THE DOUBLY CONNECTED REGION.

There are certain well-known facts for the case where the number of holes of T_n is two which we will deduce as briefly as possible in this section. If, in this case, we continue a basis (y_1, y_2) of a binary family along a fundamental path about the hole a in the positive direction, we shall have the generating substitution

$$(\bar{y}_1, \bar{y}_2) = \begin{pmatrix} m_1 & n_1 \\ m_2 & n_2 \end{pmatrix} (y_1, y_2),$$

---

[1] "Beitrage," etc. Werke, pp. 70–71.

where, in general, none of the coefficients vanish. But we can always choose bases which have a simpler generating substitution. Consider the *characteristic equation*

$$\begin{vmatrix} m_1 - \rho & n_1 \\ m_2 & n_2 - \rho \end{vmatrix} = 0. \tag{2}$$

This equation is readily shown to be invariant under any linear substitution that may be performed upon (y_1, y_2), so that it is the same for every basis, and its properties are therefore characteristic of the family. Designate by ρ_a', ρ_a'', the roots of (2); these we call the *multipliers* of the family for the hole a, the reason for this terminology appearing presently. Neither of these multipliers can vanish, since $m_1 n_2 - m_2 n_1 \neq 0$.

Two cases present themselves: —

I. $\rho_a' \neq \rho_a''$. This we designate the *ordinary case.*

Let

$$\begin{aligned} y_a' &= (\rho_a' - n_2)\, y_1 + n_1 y_2, \\ y_a'' &= m_2 y_1 + (\rho_a'' - m_1)\, y_2, \end{aligned} \tag{3}$$

it being understood that if either n_1 or m_2 vanishes, we take $\rho_a' = m_1$, $\rho_a'' = n_2$. With this convention it is easily seen that y_a', y_a'' form a basis. It may be readily verified for this basis that its generating substitution for the hole a is

$$(\bar{y}_a', \bar{y}_a'') = \begin{pmatrix} \rho_a' & 0 \\ 0 & \rho_a'' \end{pmatrix} (y_a', y_a'').$$

Thus in this case we have always a basis (y_a', y_a'') which undergoes a *multiplicative substitution* when continued along a fundamental path about a; hence the use of the term multipliers for ρ_a', ρ_a''. If k', k'', are any constants $\neq 0$, the basis $(k'y_a', k''y_a'')$ undergoes the same generating substitution. This, however, is true of no other bases, as may readily be deduced from the fact that the characteristic equation (2) is invariant. We shall call branches of the form $k'y_a'$, $k''y_a''$, *fundamental branches* for the hole a, and a basis formed from them a *fundamental basis.*

II. $\rho_a' = \rho_a''$. Here if m_2 and n_1 do not both vanish, a case to be considered presently, (3) gives but one branch and its multiples $\neq 0$ which undergo the generating substitution for the hole a,

$$\bar{y} = \rho_a' \, y'.$$

This branch we will designate by y_a', and by y_a'' we will denote any branch not a constant multiple of y_a'. Then a basis (y_a', y_a'') undergoes the generating substitution about a,

$$(\bar{y}_a', \bar{y}_a'') = \begin{pmatrix} \rho_a' & 0 \\ m & n \end{pmatrix} (y_a', y_a'').$$

From the invariance of equation (2), it follows that $n = \rho_a'$. m, however, will either assume any given value except zero, according to our choice of a basis, or else vanish for all bases. If $m \neq 0$, we shall speak of the hole a as *logarithmic* for the family; if $m = 0$, a will be called *semisingular*.[1] (y_a', y_a'') is again called a fundamental basis, but we should note that by this definition this term is, in the case before us, to be applied to any basis whatever in which the first member is a branch undergoing a multiplicative generating substitution about a. In case a is semisingular, all members of the family have the generating substitution about a,

$$\bar{y} = \rho_a' y,$$

so that every basis is fundamental; but in the logarithmic case, a particular branch y_a' and its constant multiples are the only members undergoing such a substitution about a.

If $m_2 = n_1 = 0$, (3) is invalid, but we have evidently the semisingular case.

If α' and α'' are any quantities that satisfy the relations

$$e^{2\pi i\alpha'} = \rho_a', \quad e^{2\pi i\alpha''} = \rho_a'',$$

while a_1 is some point of the hole a, we shall have, provided a_1 is not the point infinity,

$$\begin{aligned} y_a' &= (x - a_1)^{\alpha'} \phi_a'(x), \\ y_a'' &= (x - a_1)^{\alpha''} \phi_a''(x) + \frac{C_a}{2\pi i} y_a' \log(x - a_1), \end{aligned} \tag{4}$$

where $\phi_a'(x)$ and $\phi_a''(x)$ are single valued and analytic in T_2. If a_1 is the point infinity, $x - a_1$ is to be replaced by $\frac{1}{x}$ in (4). *Formula* (4) *will represent a fundamental basis in all cases;* for the ordinary and semisingular cases $C_a = 0$, while for the logarithmic case $C_a \neq 0$.

In particular, the hole a may be a single point. In this case we have Laurent's developments for ϕ_a' and ϕ_a'' in a properly chosen circle about a, and these developments may have, for certain families, only a finite number of terms with negative exponents (positive exponents if a is the point infinity). a is then said to be a *regular singular point* of the family.

§ 5. FUNDAMENTAL BASES FOR AN n-TUPLY CONNECTED REGION. CONNECTING FORMULAE.

Passing now to the general case of n holes, we see that if we join together by $n - 2$ cross-cuts all the holes except i, leaving that hole free, we have a doubly connected region, so that from § 4 we deduce the existence of fundamental bases (y_i', y_i'')

---

[1] These terms are used by Klein, as well as lengthier ones which are less convenient.

and the pair of multipliers ρ_l', ρ_l''. Proceeding in the same way for each hole, we have for $a, b, c, \ldots, l$, fundamental bases (y_a', y_a''), (y_b', y_b''), (y_c', y_c''), $\ldots$, (y_l', y_l''), and corresponding pairs of multipliers ρ_a', ρ_a''; ρ_b', ρ_b''; ρ_c', ρ_c''; $\ldots$; ρ_l', ρ_l''. A given hole may be *ordinary, logarithmic*, or *semisingular;* in each case there are certain degrees of freedom in the choice of a fundamental basis, as noted in § 4. Every fundamental basis for a can be put into the form (4), where $\phi_a'(x)$ and $\phi_a''(x)$ are single valued and analytic in any doubly connected region we may form from T_n by suitable cross-cuts which do not meet the boundary of a. Similarly for each of the other holes, with an appropriate change of notation in (4).

Let us note here an important relation between the multipliers of a family. From equation (1) we deduce at once the equation

$$\operatorname{Det} S_a \cdot \operatorname{Det} S_b \cdot \operatorname{Det} S_c \ \ldots\ \operatorname{Det} S_l = 1,$$

where Det S stands for the determinant of the substitution S. But from the invariance of (2) it follows that for every generating substitution about a we have

$$\operatorname{Det} S_a = \rho_a' \rho_a'',$$

and analogously for Det S_b, Det S_c, $\ldots$, Det S_l. Hence we have the relation

$$\rho_a' \rho_a'' \rho_b' \rho_b'' \rho_c' \rho_c'' \ \ldots\ \rho_l' \rho_l'' = 1. \tag{5}$$

By definition (2) of § 2, any set of fundamental bases is connected by a relation

$$\begin{aligned} y_a' &= \alpha y_b' + \beta y_b'' = \alpha_1 y_c' + \beta_1 y_c'' = \ldots = \alpha_{n-2} y_l' + \beta_{n-2} y_l'', \\ y_a'' &= \gamma y_b' + \delta y_b'' = \gamma_1 y_c' + \delta_1 y_c'' = \ldots = \gamma_{n-2} y_l' + \delta_{n-2} y_l''. \end{aligned} \tag{6}$$

This we will call a *connecting formula.*[1] Obviously, from the degree of arbitrariness that enters into the choice of fundamental bases, some of the coefficients in a connecting formula will be more or less at our disposal. One of the chief questions which will concern us is the determination of the remaining coefficients after those at our disposal have been fixed by a certain choice of bases in (6). As a matter of fact we shall see later that the pre-assigning of the multipliers will, in general, though not in all special cases, determine these remaining coefficients when $n = 3$. It is known that in general for $n > 3$ additional conditions are needed.

The importance of the connecting formula is evident from the following theorem, which is at once seen to be true: —

The group of a family is completely determined in case we know its multipliers, the coefficients of a connecting formula, and the constants C_a, C_b, C_c, $\ldots$, C_l, *which enter into the generating substitutions of the bases in that connecting formula.*

---

[1] Klein: Zusammenhangsformel.

§ 6. KINDRED FAMILIES.

There are two views which we may take of the group of a family; we may look upon it as an abstract group in which the members are any collection of elements obeying certain formal laws, or as a concrete group whose members are linear substitutions which a given basis undergoes, these substitutions having coefficients which are uniquely determined when the path of continuation is given. It is from this latter point of view that we proceed to study the group. We shall suppose the holes $a, b, c, \ldots, l$ always given, and limit our discussion to families having the same holes.

It is entirely conceivable that we may have two binary families with the same group; hypergeometric families, which we shall study in Part II, readily furnish us with examples of this. But first, — exactly what is meant by *families with the same group?* To answer this briefly, — *two binary families belonging to* T_n *are said to have the same group if it is possible to choose from the one family a basis* (y', y''), *and from the other a basis* $(\bar{y}', \bar{y}'')$,[1] *such that when continued together over the same path, no matter what path in* T_n *that may be, the latter basis undergoes always the same substitution as the former.* Such families we shall call *kindred families;* (y', y'') and $(\bar{y}', \bar{y}'')$ are *corresponding bases.* No matter what values we give to the constants m_1, n_1, m_2, n_2, provided $m_1 n_2 - m_2 n_1 \neq 0$, the bases

$$\begin{pmatrix} m_1 & n_1 \\ m_2 & n_2 \end{pmatrix} (y', y'') = S\,(y', y''),$$

and $S\,(\bar{y}', \bar{y}'')$ correspond. Hence:

I. *If two binary families are kindred, to every basis of the one corresponds a basis of the other,* i. e., *a necessary as well as sufficient condition that two binary families be kindred is that for every basis of the one there exist a corresponding basis of the other.*

We should note the vagueness of the word "corresponding" as used here. Obviously the same group may belong to an infinite number of bases of the same family (for example, the constant multiples of a given basis), so that this correspondence is not of a finite number of bases to a finite number.

A second theorem of great importance, whose truth is immediately obvious, is the following:

II. *A necessary and sufficient condition that a basis of one binary family correspond to a basis of another such family is that the two bases have the same generating substitutions.*

---

[1] This is not to be confused with a notation used in §§ 3 and 4.

From the theorem stated at the end of § 5, we see that Theorem II can be given in terms of the connecting formula; in fact, as we shall proceed to prove, the following gives us a criterion that two families be kindred:

III. *In order that two binary families belonging to* $\mathrm{T_n}$ *be kindred, it is both necessary and sufficient that the families have the same multipliers, and connecting formulae in which the corresponding coefficients and the associated constants* $\mathrm{C_a}$, $\mathrm{C_b}$, $\mathrm{C_c}$, . . ., $\mathrm{C_l}$, *are the same.*

From the preceding theorems the sufficiency of this condition is obvious. To show its necessity, we first note the fact that if a fundamental basis for a given hole with multipliers ρ', ρ'', has at that hole the generating substitution $\begin{pmatrix} \rho' & 0 \\ \rho' C & \rho'' \end{pmatrix}$, a corresponding basis of the other family must, by definition, have the same generating substitution at that hole. Forming the characteristic equation (2), we see that the multipliers of the second family at the given hole are also ρ', ρ'', as was to be proved. Moreover, the possession of the generating substitution $\begin{pmatrix} \rho' & 0 \\ \rho' C & \rho'' \end{pmatrix}$ about a hole for which the multipliers are ρ', ρ'', characterizes a fundamental basis at that hole. We have thus proved that if (y', y'') is a basis of the one family fundamental for a given hole, every corresponding basis of the other family must be fundamental for the same hole, and must besides have the same associated constant C as (y', y'').

Let (y_a', y_a'') and $(\bar{y}_a', \bar{y}_a'')$ be corresponding bases of the two families; a connecting formula of the first family is

$$(y_a', y_a'') = B\,(y_b', y_b'') = C\,(y_c', y_c'') = \ldots = L\,(y_l', y_l''). \tag{7}$$

Denoting by B^{-1} the inverse of B, we have

$$\begin{aligned} B^{-1}\,(y_a', y_a'') &= (y_b', y_b''), \\ C^{-1}\,(y_a', y_a'') &= (y_c', y_c''), \\ &\cdots\cdots\cdots \\ &\cdots\cdots\cdots \\ L^{-1}\,(y_a', y_a'') &= (y_l', y_l''). \end{aligned}$$

Since the basis $B^{-1}\,(\bar{y}_a', \bar{y}_a'')$ must correspond to the basis $B^{-1}\,(y_a', y_a'')$, it follows from what we have just seen that $B^{-1}\,(\bar{y}_a', \bar{y}_a'')$ is a basis $(\bar{y}_b', \bar{y}_b'')$ fundamental at b and having the same associated constant C_b as (y_b', y_b''). Proceeding similarly, we have

$$\begin{aligned} C^{-1}\,(\bar{y}_a', \bar{y}_a'') &= (\bar{y}_c', \bar{y}_c''), \\ &\cdots\cdots\cdots \\ &\cdots\cdots\cdots \\ L^{-1}\,(\bar{y}_a', \bar{y}_a'') &= (\bar{y}_l', \bar{y}_l''), \end{aligned}$$

so that for the second family we have a connecting formula,

$$(\bar{y}_a', \bar{y}_a'') = B\,(\bar{y}_b', \bar{y}_b'') = C\,(\bar{y}_c', \bar{y}_c'') = \dots = L\,(\bar{y}_l', \bar{y}_l''), \tag{8}$$

which is the same as (6), including the associated constants C_a, C_b, C_c, . . ., C_l. Our proof is thus completed.

A generalization of this result which we shall need in the next section, and whose proof is readily supplied from the proof of Theorem III, is:

IV. *If* (y', y'') *and* $(\bar{y}', \bar{y}'')$ *are any two corresponding bases of two kindred binary families, and if* (y', y'') *is related to fundamental bases of its family by the formula*

$$(y', y'') = A'\,(y_a', y_a'') = B'\,(y_b', y_b'') = \dots = L'\,(y_l', y_l''),$$

then $(\bar{y}', \bar{y}'')$ *is also related to properly chosen fundamental bases of its family by the formula*

$$(\bar{y}', \bar{y}'') = A'\,(\bar{y}_a', \bar{y}_a'') = B'\,(\bar{y}_b', \bar{y}_b'') = \dots = L'\,(\bar{y}_l', \bar{y}_l''),$$

the associated constants C_a, C_b, C_c, . . ., C_l, *as well as the coefficients, being the same in both formulae.*

§ 7. RELATIONS BETWEEN KINDRED BINARY FUNCTIONS.

Before proceeding, it will be convenient to introduce a few new terms. If (y', y'') and $(\bar{y}', \bar{y}'')$ are corresponding bases of two kindred binary families, then we will speak of y' and $\bar{y}'$, and also of y'' and $\bar{y}''$, as *corresponding branches.*

We will call functions whose branches are members of a binary family, *binary functions;* and by two *kindred binary functions,* we mean functions two of whose branches are corresponding branches of two kindred binary families.

We proceed now to develop certain relations between corresponding branches belonging to kindred binary families; but by the principle of the permanence of functional relations, the equations developed will be in fact relations between the kindred binary functions to which those branches belong.

Let (y', y'') and $(\bar{y}', \bar{y}'')$ be corresponding bases of two kindred families; using the notation of Theorem IV, § 6, we have

$$\begin{vmatrix} y' & \bar{y}' \\ y'' & \bar{y}'' \end{vmatrix} = \text{Det } A' \cdot \begin{vmatrix} y_a' & \bar{y}_a' \\ y_a'' & \bar{y}_a'' \end{vmatrix} = \text{Det } B' \cdot \begin{vmatrix} y_b' & \bar{y}_b' \\ y_b'' & \bar{y}_b'' \end{vmatrix} = \dots = \text{Det } L' \cdot \begin{vmatrix} y_l' & \bar{y}_l' \\ y_l'' & \bar{y}_l'' \end{vmatrix}.$$

From these equations it follows that the determinant

$$\begin{vmatrix} y' & \bar{y}' \\ y'' & \bar{y}'' \end{vmatrix}$$

defines a function undergoing only multiplicative substitutions for every path in T_n; for, using the notation of (4), we have

$$\begin{vmatrix} y_a{}' & \bar{y}_a{}' \\ y_a{}'' & \bar{y}_a{}'' \end{vmatrix}$$

$$= \begin{vmatrix} (x-a_1)^{\alpha'}\phi_a{}'(x) & (x-a_1)^{\bar{\alpha}'}\bar{\phi}_a{}'(x) \\ (x-a_1)^{\alpha''}\phi_a{}''(x)+\frac{C_a}{2\pi i}(x-a_1)^{\alpha'}\phi_a{}'(x)\log(x-a_1) & (x-a_1)^{\bar{\alpha}''}\bar{\phi}_a{}''(x)+\frac{C_a}{2\pi i}(x-a_1)^{\bar{\alpha}'}\bar{\phi}_a{}'(x)\log(x-a_1) \end{vmatrix}$$

$$= \begin{vmatrix} (x-a_1)^{\alpha'}\phi_a{}'(x) & (x-a_1)^{\bar{\alpha}'}\bar{\phi}_a{}'(x) \\ (x-a_1)^{\alpha''}\phi_a{}''(x) & (x-a_1)^{\bar{\alpha}''}\bar{\phi}_a{}''(x) \end{vmatrix}.$$

But since the multipliers of kindred families are the same, $\alpha' - \bar{\alpha}'$ and $\alpha'' - \bar{\alpha}''$ must both be integers, so that we may write

$$\begin{vmatrix} y_a{}' & \bar{y}_a{}' \\ y_a{}'' & \bar{y}_a{}'' \end{vmatrix} = (x-a_1)^{\alpha'+\bar{\alpha}''}\phi_a(x),$$

where $\phi_a(x)$ is single valued and analytic in a ring about a; similarly for the other holes. Hence we have the equation

$$(9)\qquad \begin{vmatrix} y' & \bar{y}' \\ y'' & \bar{y}'' \end{vmatrix} = (x-a_1)^{\alpha'+\bar{\alpha}''}(x-b_1)^{\beta'+\bar{\beta}''}\ldots(x-l_1)^{\lambda'+\bar{\lambda}''}\phi(x),^{1}$$

where $\phi(x)$ is single valued and analytic everywhere in T_n, except possibly at the point infinity, which, in (9), is supposed not to be an isolated singular point. If it is, it is easy to see how this discussion is to be modified. Equation (5) gives the relation

$$\alpha' + \bar{\alpha}'' + \beta' + \bar{\beta}'' + \ldots + \lambda' + \bar{\lambda}'' = \text{an integer},$$

so that $\phi(x)$ has at most a pole in the point infinity, and must be single valued there. In particular, since the exponents concerned may be increased or diminished by any integers, we may so choose them that $\phi(x)$ will be analytic in the point infinity.

Between any three kindred functions there exists a relation which we may now easily deduce. Let (y', y''), $(\bar{y}', \bar{y}'')$, $y', \bar{\bar{y}}'')$, be corresponding bases of three kindred binary families, and let y, $\bar{y}$, $\bar{\bar{y}}$, be any three corresponding branches of the respective families. The bases chosen may always be so taken that if $y = \alpha y' + \beta y''$, we shall have

$$\bar{y} = \alpha\bar{y}' + \beta\bar{y}'', \quad \bar{\bar{y}} = \alpha\bar{\bar{y}}' + \beta\bar{\bar{y}}''.$$

Having so chosen the bases that these relations hold, we obtain the equation

$$0 = \begin{vmatrix} y & \bar{y} & \bar{\bar{y}} \\ y' & \bar{y}' & \bar{\bar{y}}' \\ y'' & \bar{y}'' & \bar{\bar{y}}'' \end{vmatrix} = y\begin{vmatrix} \bar{y}' & \bar{\bar{y}}' \\ \bar{y}'' & \bar{y}'' \end{vmatrix} - \bar{y}\begin{vmatrix} y' & \bar{\bar{y}}' \\ y'' & \bar{\bar{y}}'' \end{vmatrix} + \bar{\bar{y}}\begin{vmatrix} y' & y' \\ y'' & \bar{y}'' \end{vmatrix}.$$

The coefficients of y, y, $\bar{\bar{y}}$, here have a form analogous to (9), so that, since the multipliers of kindred families are the same, we have only to divide through by suitable powers of $(x - a_1)$, $(x - b_1)$, . . ., $(x - l_1)$, to obtain the relation

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[1] The notation here is readily understood by comparison with that used for the hole a.

$$y\,\Phi(x) + \bar{y}\,\bar{\Phi}(x) + \bar{\bar{y}}\,\bar{\bar{\Phi}}(x) = 0, \tag{10}$$

where Φ, $\bar{\Phi}$, $\bar{\bar{\Phi}}$ are single valued and analytic throughout T_n. Hence we may state the theorem:

I. *Any three kindred binary functions are connected by a linear and homogeneous relation whose coefficients are functions of the independent variable single valued and analytic throughout* T_n.

In case all the holes are *regular singular points* (as defined in § 4), Φ, $\bar{\Phi}$, $\bar{\bar{\Phi}}$, may be taken polynomials;[1] it will be sufficient here to refer to the more explicit formulæ of Part II for the case $n = 3$, from which it is easy to see how to pass to the general case $n > 3$.

As a particular application of Theorem I we may note that any binary function and any derivative of that function are kindred, as can easily be verified from Theorem III of § 6. Hence from Theorem I alone we can deduce the differential equation

$$\phi_1 \frac{d^2 y}{d x^2} + \phi_2 \frac{d y}{d x} + \phi_3\, y = 0, \tag{11}$$

in which the coefficients are single valued and analytic throughout T_n. If all the holes are regular singular points, these coefficients will be rational, and hence can be taken polynomials. Equation (11) is satisfied by every member of the binary family to which the function y belongs. Conversely, we know from the theory of differential equations that all the solutions of such an equation constitute a binary family. Hence we have the theorem:

II. *A binary family consists of all the solutions of a differential equation* (11); *and every equation* (11) *defines a binary family.*

B. CLASSIFICATION OF BINARY FAMILIES FOR THE CASE $n = 3$.

§ 1. THE PROBLEM.

The question — when are two binary families kindred? — has been answered in § 6 of Part I, A, by Theorem III, and it is to be noted that this answer has two parts. The first requirement is that the two families have the same multipliers, the second that they have a common connecting formula. To be sure we shall find cases where the satisfying of the first condition entails as a necessary consequence the satisfying of the second, but this is by no means always true.

Can we so classify binary families for the case $n = 3$ that two families belonging to the same class will have the same group when their multipliers are the same, but will always have different groups when they belong to different classes? This is the question which will now occupy us, and which will be answered affirmatively by the actual construction of such a classification.

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[1] See Riemann's paper, Zwei allgemeine Sätze, etc. Werke, p. 379.

It is instructive to observe how this may be done for the case $n = 2$. Here families with the same multipliers are kindred unless the two multipliers of a hole a are equal to each other; in this case both holes may be logarithmic for some families, and semisingular for others. It will readily be seen that the following is a classification of the kind desired:

$$\text{Type I.}\quad \rho_a' \neq \rho_a''.$$

$$\text{Type II.}\quad \rho_a' = \rho_a'' \begin{cases} \text{Class 1.} & C_a = 0, \\ \text{Class 2.} & C_a \neq 0. \end{cases}$$

§ 2. THE FOUR TYPES.

Obviously the study of connecting formulæ will be most essential to our work; but it is important to notice that we need concern ourselves only with the part

$$\begin{aligned} y_a' &= \alpha\, y_b' + \beta\, y_b'', \\ y_a'' &= \gamma\, y_b' + \delta\, y_b'', \end{aligned} \tag{12}$$

since the relation $S_c\, S_b\, S_a = 1$, noted on page 9, shows that the generating substitution for c is expressible in terms of those for a and b. But this relation $S_c\, S_b\, S_a = 1$ gives, besides the equation

$$\rho_a' \rho_a'' \rho_b' \rho_b'' \rho_c' \rho_c'' = 1, \tag{13}$$

already given for the general case on page 12, an additional equation in terms of α, β, γ, δ, C_a, C_b, and the multipliers, which is most important for our subsequent work. We obtain this as follows: If we turn back to § 4 of Part I, A, we see that ρ_c' and ρ_c'' are roots of the invariant characteristic equation

$$\begin{vmatrix} m_1 - \rho & n_1 \\ m_2 & n_2 - \rho \end{vmatrix} = 0, \tag{14}$$

where $\begin{pmatrix} m_1 & n_1 \\ m_2 & n_2 \end{pmatrix}$ is the generating substitution of any basis about c; if the inverse of such a generating substitution, the roots are $\frac{1}{\rho_c'}$, $\frac{1}{\rho_c''}$. On this latter hypothesis (14) gives us two invariants. The first is

$$\frac{1}{\rho_c' \rho_c''} = \begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix},$$

but this is readily seen to be (13) in another form.[1] The other invariant is

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[1] $m_1 n_2 - m_2 n_1$ is in fact the product Det S_a • Det S_b. But Det $S_a = \rho_a' \rho_a''$, and Det $S_b = \rho_b' \rho_b''$.

$$\frac{1}{\rho_c'} + \frac{1}{\rho_c''} = m_1 + n_2, \tag{15}$$

and this equation will be found most essential for the classification we wish to make.

To obtain the values of m_1 and n_2, let us use the relation

$$S_c^{-1} = \begin{pmatrix} m_1 & n_1 \\ m_2 & n_2 \end{pmatrix} = S_b\, S_a.$$

From formula (12) we can easily compute $S_b\, S_a$ for the basis (y_b', y_b''), and in this way we find

$$m_1 = \rho_a' \rho_b' [\delta - C_a \beta] \frac{a}{\Delta} - \rho_a'' \rho_b' \frac{\beta\gamma}{\Delta},$$

$$n_2 = [\rho_a' \rho_b'' (-\gamma + C_a a) + \rho_a' \rho_b' C_b (\delta - C_a \beta)] \frac{\beta}{\Delta} + [\rho_a'' \rho_b'' a - \rho_a'' \rho_b' C_b \beta] \frac{\delta}{\Delta},$$

where Δ stands for the non-vanishing determinant $a\delta - \beta\gamma$. If we multiply both sides of (15) by Δ, and perform a few simple transformations, we obtain the equation

$$\begin{aligned} a\delta \frac{1}{\rho_a' \rho_b' \rho_c' \rho_c''} (\rho_a' \rho\,' \rho_c' - 1)(\rho_a' \rho_b' \rho_c'' - 1) - C_a a \beta \rho_a' (\rho_b' - \rho_b'') \\ + \beta\gamma \frac{1}{\rho_c'} (\rho_a'' \rho_b' \rho_c' - 1)(\rho_a' \rho_b'' \rho_c' - 1) + C_b \beta \delta \rho_b' (\rho_a' - \rho_a'') + C_a C_b \beta^2 \rho_a' \rho_b' = 0. \end{aligned} \tag{16}$$

The presence of products of multipliers $\rho_a' \rho_b' \rho_c'$, $\rho_a' \rho_b' \rho_c''$, etc., in this equation gives us a hint that an important principle of classification is introduced when we state, for a given family, whether such a product is, or is not, equal to unity; such knowledge is, in fact, essential for the discussion of (16). That this has a deeper significance will be manifest when we have observed subsequently that whenever such a product is equal to unity, there exists a member of the family which undergoes only multiplicative substitutions when continued along any closed path in T_3. Accordingly, we divide all binary families having the same three holes into four *types*, characterized as follows:

Type	I.	$\rho_a' \rho_b' \rho_c' \neq 1,$	$\rho_a'' \rho_b' \rho_c' \neq 1,$	$\rho_a' \rho_b'' \rho_c' \neq 1,$	$\rho_a' \rho_b' \rho_c'' \neq 1.$
"	II.	$\rho_a' \rho_b' \rho_c' = 1,$	$\rho_a'' \rho_b' \rho_c' \neq 1,$	$\rho_a' \rho_b'' \rho_c' \neq 1,$	$\rho_a' \rho_b' \rho_c'' \neq 1.$
"	III.	$\rho_a' \rho_b' \rho_c' = 1,$	$\rho_a'' \rho_b' \rho_c' = 1,$	$\rho_a' \rho_b'' \rho_c' \neq 1,$	$\rho_a' \rho\,' \rho_c'' \neq 1.$
"	IV.	$\rho_a' \rho_b' \rho_c' = 1,$	$\rho_a'' \rho_b' \rho_c' = 1,$	$\rho_a' \rho_b'' \rho_c' = 1,$	$\rho_a' \rho_b' \rho_c'' = 1.$

In each type the behavior of the remaining four products of multipliers is determined by (13). It might seem at first glance that there should be another type characterized by the relations

$$\rho_a'\rho_b'\rho_c' = 1, \quad \rho_a''\rho_b'\rho_c' = 1, \quad \rho_a'\rho_b''\rho_c' = 1, \quad \rho_a'\rho_b'\rho_c'' \neq 1;$$

but these four equations are inconsistent, since from the first three follows

$$\rho_a'\rho_b'\rho_c'' = 1.$$

By a proper choice of notation every binary family in T_3 *can be brought under one, and but one, of the above four types.*

§ 3. CLASSIFICATION OF FAMILIES UNDER EACH TYPE.

The types have been determined by means of characteristic relations among the multipliers. In classifying under the types we make use also of the second condition that two families be kindred, namely, the requirement that they have a common connecting formula. As we have already remarked, we need use only the part (12) in this discussion. We proceed as follows:

Suppose we are given a formula (12) for a given family; and suppose also we are given a second family with the same multipliers, and belonging to it a formula

$$\begin{aligned} Y_a' &= \bar{\alpha}\, Y_b' + \bar{\beta}\, Y_b'', \\ Y_a'' &= \bar{\gamma}\, Y_b' + \bar{\delta}\, Y_b''; \end{aligned} \tag{17}$$

then, according to our criterion, *the two families are kindred when and only when, by a new choice of fundamental bases in the second family,* (17) *may be changed to*

$$\begin{aligned} \bar{y}_a' &= \alpha\, \bar{y}_b' + \beta\, \bar{y}_b'', \\ \bar{y}_a'' &= \gamma\, \bar{y}_b' + \delta\, \bar{y}_b'', \end{aligned} \tag{18}$$

where the associated constants C, *as well as the coefficients, have the same value as in* (12). *If this can be done, the two families, which are of the same type since they have the same multipliers, will belong to the same* CLASS; *if not, to different classes.* Here, then, is given the basis of our further classification. Let us observe that this is a classification of families by means of their groups, but it affords a classification of the groups as well.

To see how a formula (17) may be changed, let us recollect the degree of arbitrariness which enters into our choice of fundamental bases; we may multiply each member of a fundamental basis by any constant ≠ 0, and still have a fundamental basis; moreover, if the hole is ordinary, this is the only way in which we can pass from one such basis to another.

We may illustrate our course of procedure by considering the case where both a and b are ordinary, while none of the coefficients in (12) and (17) vanish. We can here write, instead of (17), the new formula

$$(p\,Y_a') = \frac{\bar{\alpha}\,p}{r}(r\,Y_b') + \frac{\bar{\beta}\,p}{s}(s\,Y_b''),$$
$$(q\,Y_a'') = \frac{\bar{\gamma}\,q}{r}(r\,Y_b') + \frac{\bar{\delta}\,q}{s}(s\,Y_b'').$$

We can always choose the constants p, q, r, s, so as to satisfy the equations

$$\frac{\bar{\alpha}\,p}{r} = \alpha, \quad \frac{\bar{\beta}\,p}{s} = \beta, \quad \frac{\bar{\delta}\,q}{s} = \delta,$$

and we shall then have

$$\frac{q}{r} = \frac{\delta\,\alpha\,\bar{\beta}}{\bar{\delta}\,\bar{\alpha}\,\beta}.$$

Our new formula which we now substitute for (17) is

$$\bar{y}_a = \alpha\,\bar{y}_b' + \beta\,\bar{y}_b'',$$
$$\bar{y}_a'' = \bar{\gamma}\frac{\delta\,\alpha\,\bar{\beta}}{\bar{\delta}\,\bar{\alpha}\,\beta}\bar{y}_b' + \delta\,\bar{y}_b'',$$

where $(\bar{y}_a', \bar{y}_a'')$, $(\bar{y}_b', \bar{y}_b'')$, stand for the fundamental bases $(p\,Y_a', q\,Y_a'')$, $(r\,Y_b', s\,Y_b'')$, respectively. Obviously a necessary and sufficient condition that the two families (which have the same multipliers, and for both of which $C_a = C_b = 0$) be kindred is

$$\bar{\gamma}\frac{\delta\,\alpha\,\bar{\beta}}{\bar{\delta}\,\bar{\alpha}\,\beta} = \gamma.$$

And to find out whether this is true, we need only make use of equation (16); but this we defer for the present.

The work in this case has been given at some length in order to make the plan of procedure clear. If a hole is semisingular, we must remember that any basis whatever is fundamental at that hole; if logarithmic, we may change from one fundamental basis to another by taking any multiple of the first member of the old for the first member of the new, the second member of the new basis being any linearly independent member of the family; but the associated constant C will in general suffer a change.

So much, by way of generalities, will suffice. The method of classification will become clearer as we proceed, case by case, in the following pages. The general procedure is to observe for each type the possible forms of (12), for families with the same multipliers, which cannot be carried over into each other by a change of fundamental bases; to each such form will belong a *class* of families. Such a plan of classification is evidently of the sort we wish; for families with the same multipliers will be kindred when, and only when, they belong to the same class.

We now proceed to an examination of each type.

Type I. $\rho_a' \rho_b' \rho_c' \neq 1, \quad \rho_a'' \rho_b' \rho_c' \neq 1, \quad \rho_a' \rho_b'' \rho_c' \neq 1, \quad \rho_a' \rho_b' \rho_c'' \neq 1.$

We observe first that the coefficient β in (12) cannot vanish for this type; for if $\beta = 0$, all the terms of (16) vanish except the first, which cannot vanish, since no product of multipliers in it $= 1$, and $\alpha\delta - \beta\gamma \neq 0$, so that $\alpha\delta \neq 0$. As a consequence, we see that neither a nor b can be semisingular, for any basis is fundamental for a semisingular hole, so that if a or b were semisingular we could write $y_a' = y_b'$, and this would be the first line of a possible formula (12); but here $\beta = 0$, and this we have seen to be impossible. Whether we use b or c in this work is only a matter of notation, so that it is true for this type that no hole is semisingular. Three possibilities now present themselves.

If the two holes a and b are ordinary, we have $C_a = C_b = 0$, and, using the two facts that $\alpha\delta - \beta\gamma \neq 0$, and that no product of multipliers $= 1$, we may easily assure ourselves, from (16), that $\alpha, \beta, \gamma, \delta$ can none of them vanish. On page 21 we have seen that α, β, δ can be taken at pleasure, but $\neq 0$; (16) then uniquely determines γ. Hence in this case all families having the same multipliers have a common formula (12), and hence are kindred.

If a is logarithmic, while b is ordinary, we have $\rho_a' = \rho_a''$, $C_a \neq 0$, $C_b = 0$. Using again the fact that $\alpha\delta - \beta\gamma \neq 0$, so that if α were to vanish γ could not, we deduce from (16) that α cannot vanish. We see here, as on page 21, that α and β are at our choice, except that neither can vanish. Fixing y_a', we can choose y_a'' so that C_a has any preassigned value $\neq 0$. Form now a new y_a'', by adding to the old such a multiple of y_a' that in the new formula $\delta = 0$; C_a remains unaltered. In this new formula, then, C_a, α, and β are arbitrary, while $\delta = 0$; γ is uniquely determined by (16). Hence all families here have a common formula (12), providing they have the same multipliers.

If both a and b are logarithmic, we may proceed as follows: Choose bases (y_a', y_a''), (y_b', y_b''), so that C_a and C_b have any preassigned values $\neq 0$. Take a new y_b'', by adding to the old such a multiple of y_b' that $\alpha = 0$; take as a new y_a' and y_a'' such a multiple of the old pair that β will take on an arbitrary value $\neq 0$, C_a and C_b retaining their former values. Finally, for a last y_a'', add to the former such a multiple of y_a' that the new $\delta = 0$. These changes leave C_a and C_b still unaltered. Then γ, for this new formula, is uniquely determined by (16). Again we have, for this case, a formula possessed by every family with the same multipliers.

We have thus exhausted all cases; a family with given multipliers belongs under one and only one of these cases, under each of which families with the same multipliers are always kindred. Hence *Type I furnishes but one class.* Stated in other words: *under Type I the group is always completely determined by the multipliers.*

Type II. $\rho_a'\rho_b'\rho_c'=1,\quad \rho_a''\rho_b'\rho_c'\neq 1,\quad \rho_a'\rho_b''\rho_c'\neq 1,\quad \rho_a'\rho_b'\rho_c''\neq 1.$

Every hole is here ordinary; if, for instance, $\rho_a'=\rho_a''$, we should have $\rho_a''\rho_b'\rho_c'=1$, contrary to hypothesis. Equation (16) reduces to $\beta\gamma=0$. We have, then, the following three cases:

1. $\beta=0,\quad \gamma\neq 0.$
2. $\beta\neq 0,\quad \gamma=0.$
3. $\beta=0,\quad \gamma=0.$

In all these cases neither α nor δ can vanish, since $\alpha\delta-\beta\gamma\neq 0$. In each case, the coefficients of (12) that do not vanish can take on any value $\neq 0$ by an appropriate choice of fundamental bases, but it is impossible to prevent the specified coefficients from vanishing, or to make any others vanish. Every connecting formula can be reduced to one of *three* special ones, marked by the respective cases, but these three cannot be carried over into each other. Families of Type II, with the same multipliers, may fall under any one of these three cases, so that we have here three classes; these we will designate Classes 1, 2, 3, respectively.

Type III. $\rho_a'\rho_b'\rho_c'=1,\quad \rho_a''\rho_b'\rho_c'=1,\quad \rho_a'\rho_b''\rho_c'\neq 1,\quad \rho_a'\rho_b'\rho_c''\neq 1.$

Here $\rho_a'=\rho_a''$, but $\rho_b'\neq\rho_b''$, $\rho_c'\neq\rho_c''$; hence C_b vanishes, and (16) reduces to $C_a\,\alpha\,\beta=0$. Since α and β cannot both vanish, on account of the relation $\alpha\delta-\beta\gamma\neq 0$, and since when $C_a=0$ we can always take $\beta=0$, on account of the fact that (y_b', y_b'') is then fundamental at a, we have only the following three cases to consider:

1. $C_a\neq 0$, $\alpha\neq 0$. Here β always vanishes, from (16), while δ never vanishes, since $\alpha\delta-\beta\gamma\neq 0$. Given a formula (12), we may first so change it by a new choice of y_b' and y_b'' that α and δ have any values $\neq 0$ that we please. The resulting formula we can again change by using as a new y_a'' the old y_a'' (which we can suppose to have been so chosen that C_a has any value we please $\neq 0$), plus any multiple we please of y_a'. This leaves C_a, α, δ unchanged, but we can make the new γ anything we please, including zero. Hence the formulæ (12) of all families with the same multipliers under this case can be taken the same (including C_a).

2. $C_a\neq 0$, $\alpha=0$. The discussion here is much the same as for the previous case. β can never vanish, nor can γ; we can take β, γ, C_a anything we please except zero, while to δ we can give any value including zero, so that the formulæ (12) of all families under this case can be taken the same.

3. $C_a = 0$. Since every basis is fundamental at a, we can here always take for (12)

$$\begin{aligned} y_a' &= y_b', \\ y_a'' &= y_b''. \end{aligned}$$

A family of Type III, with given multipliers, may belong to any one of these three cases, and the formulæ of different cases (including the values of C_a) can obviously not be carried over into each other. Hence we have here three classes which we number in correspondence with the above three cases.

Type IV. $\rho_a'\rho\,'\rho_c' = 1, \quad \rho_a''\rho_b'\rho_c' = 1, \quad \rho_a'\rho_b''\rho_c' = 1, \quad \rho_a'\rho_b'\rho_c'' = 1.$

We may deduce from the relations characterizing this type the equations $\rho_a' = \rho_a''$, $\rho_b' = \rho_b''$, $\rho_c' = \rho_c''$. Equation (16) reduces to $C_a\, C_b\, \beta = 0$.

It will be convenient, though not necessary, to make use of C_c here; we may then consider the following five cases:

1. $C_a = 0, \quad C_b \neq 0, \quad C_c \neq 0.$
2. $C_a \neq 0, \quad C_b = 0, \quad C_c \neq 0.$
3. $C_a \neq 0, \quad C_b \neq 0, \quad C_c = 0.$
4. $C_a = 0, \quad C_b = 0, \quad C_c = 0.$
5. $C_a \neq 0, \quad C_b \neq 0, \quad C_c \neq 0.$

These exhaust all possible cases, for if two holes are semisingular, it is obvious that the third will also be semisingular. The groups corresponding to each case are evidently different. In all cases we may take for (12)

$$\begin{aligned} y_a' &= y_b', \\ y_a'' &= y_b''; \end{aligned} \tag{19}$$

for in cases 1, 2, 4 every basis is fundamental either at a or b, while in cases 3 and 5 equation (16) gives $\beta = 0$, and since b is logarithmic and $\delta \neq 0$, $\alpha \neq 0$ (on account of the relation $\alpha\delta - \beta\gamma \neq 0$), we are at liberty to rename $\alpha y_b'$ and $\gamma y_b' + \delta y_b''$, which are fundamental bases for b, y_b', and y_b'' respectively. To find whether there is but one class under each case, we must see whether in (19) C_a and C_b can be given the same set of values for every family under each case.

1. $C_a = 0$. C_b can always be given any value we please $\neq 0$, since y_b'' is at our choice. There is but one class here.

2, 3. There can obviously be but one class in each of these cases, from their resemblance to Case 1.

4. $C_a = C_b = 0$, always, in this case. There is but one class here.

5. We shall find an infinite number of classes under this case, for when we have chosen our bases in (19) so that one of the two quantities C_a, C_b is fixed, the other is

not at our disposal. To show this, let us introduce in place of the old bases (y_a', y_a''), (y_b', y_b''), new bases (Y_a', Y_a''), (Y_b', Y_b''), satisfying the equations

$$\begin{aligned} Y_a' &= Y_b', \\ Y_a'' &= Y_b'', \end{aligned}$$

the associated constants C having the values $\bar{C}_a$, $\bar{C}_b$. We must have

$$\begin{aligned} Y_a' &= \lambda_1 y_a', \\ Y_a'' &= \mu_1 y_a' + \nu_1 y_a'', \\ Y_b' &= \lambda_2 y_b', \\ Y_b'' &= \mu_2 y_b' + \nu_2 y_b''. \end{aligned}$$

By comparing with (19) we easily deduce the equations

$$\lambda_1 = \lambda_2, \quad \mu_1 = \mu_2, \quad \nu_1 = \nu_2; \quad \lambda_1 \neq 0, \quad \nu_1 \neq 0;$$

$$\frac{\bar{C}_a}{C_a} = \frac{\nu_1}{\lambda_1}, \quad \frac{\bar{C}_b}{C_b} = \frac{\nu_2}{\lambda_2},$$

giving the equation,

$$\frac{\bar{C}_a}{\bar{C}_b} = \frac{C_a}{C_b} = \kappa.$$

Hence no matter what basis we use in (19), $\kappa = \frac{C_a}{C_b}$ is the same for a given family; *i. e.* κ is an invariant of a family belonging to this case. Accordingly all families here with the same multipliers, *and for which* κ *is the same*, and only such families, have the same group. But is κ capable of taking on an infinite number of values for families with the same multipliers? We can at least see no restrictions on its value here (except that we are to avoid cases 1, 2, 3, 4), and in the last subdivision of Part II we shall give an example where κ can assume an infinite number of values. Each value of κ, then, gives a class. We get cases 1, 2, 3, by putting $\kappa = 0, \infty, -1$, respectively. In Case 4, κ has no meaning. Cases 1, 2, 3, 4 give each a class, which we number correspondingly, and we have besides these an infinite number of classes corresponding each to a different value of κ.

§ 4. TABULATION OF CLASSES.

It is for many purposes useful to know the characteristics of the complete connecting formula

$$\begin{aligned} y_a' &= \alpha y_b' + \beta y_b'' = \alpha_1 y_c' + \beta_1 y_c'', \\ y_a'' &= \gamma y_b' + \delta y_b'' = \gamma_1 y_c' + \delta_1 y_c'', \end{aligned} \tag{20}$$

for each class. We have already discussed the determination of C_a, C_b, α, β, γ, δ. To find C_c, α_1, β_1, γ_1, δ_1, or rather those which depend upon the others after we have given particular values to those at our choice, we use the four equations furnished by

$S_c\, S_b\, S_a = 1,$[1]

which we shall give in the next section. At present it is enough for our purposes to find which of the coefficients in the complete connecting formula of a family vanish, and this can easily be done with the aid of the results of the preceding section; either the symmetry of conditions at b and c, or else the relations characterizing the type, will tell us which of the coefficients α_1, β_1, γ_1, δ_1 *must* vanish, and which may be made to vanish by a proper choice of bases in (20). We already know that it must be possible to reduce all complete formulæ (20) of families of the same class with the same multipliers to one and the same formula (see Theorem III of Part I, A, § 6). In the following table we briefly characterize each class.

Type I. $\rho_a' \rho_b' \rho_c' \neq 1, \quad \rho_a'' \rho_b' \rho_c' \neq 1, \quad \rho_a' \rho_b'' \rho_c \neq 1, \quad \rho_a' \rho_b' \rho_c'' \neq 1.$

There is but one class under this type. It is always possible to take a formula (20) in which no coefficients vanish. No hole is semisingular. Using the term *everywhere fundamental branch* to designate a member of a binary family which undergoes multiplicative substitutions only for all paths in T_3, we deduce from the behavior of the connecting formulæ the fact that there can be no everywhere fundamental branches belonging to families of this type.

Type II. $\rho_a' \rho_b' \rho_c' = 1, \quad \rho_a'' \rho_b' \rho_c' \neq 1, \quad \rho_a' \rho_b'' \rho_c' \neq 1, \quad \rho_a' \rho_b' \rho_c'' \neq 1.$

In every family under this and the two remaining types there is an everywhere fundamental branch. For this type all holes are ordinary.

Class 1. $\beta = \beta_1 = 0$, while the other coefficients in (20) can never vanish. Only a branch y_a', and its constant multiples, is everywhere fundamental, its multipliers being ρ_a', ρ_b', ρ_c'.

Class 2. $\gamma = \gamma_1 = 0$; no other coefficients in (20) can vanish. Only a branch y_a'', and its constant multiples, is everywhere fundamental, its multipliers being ρ_a'', ρ_b'', ρ_c''.

Class 3. $\beta = \beta_1 = \gamma = \gamma_1 = 0$; no other coefficient in (20) can vanish. Only two linearly independent branches, y_a' and y_a'', and their constant multiples, are everywhere fundamental, their multipliers being ρ_a', ρ_b', ρ_c', and ρ_a'', ρ_b'', ρ_c'', respectively.

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[1] Equivalent to only three besides (13).

Type III. $\rho_a' \rho_b' \rho_c' = 1, \quad \rho_a'' \rho_b' \rho_c' = 1, \quad \rho_a' \rho_b'' \rho_c' \neq 1, \quad \rho_a' \rho_b' \rho_c'' \neq 1.$

Here $\rho_a' = \rho_a''$, but b and c are ordinary.

Class 1. β and β_1 must vanish, and γ can always be made to vanish, in which case none of the other coefficients in (20) = 0. a is logarithmic. There is but one branch y_a', and its constant multiples, everywhere fundamental, its multipliers being ρ_a', ρ_b', ρ_c'.

Class 2. α and α_1 must vanish, and δ can always be made to vanish, in which case none of the other coefficients in (20) = 0. a is logarithmic. Only a branch y_a', and its constant multiples, is everywhere fundamental, its multipliers being ρ_a', ρ_b'', ρ_c''.

Class 3. We may take none of coefficients in (20) zero, or as many as four; for instance β, β_1, γ, γ_1. a is semisingular. Only two linearly independent branches, y_a' and y_a'', and their constant multiples, are everywhere fundamental, their multipliers being ρ_a', ρ_b', ρ_c', and ρ_a', ρ_b'', ρ_c'', respectively.

Type IV. $\rho_a' \rho_b' \rho_c' = 1, \quad \rho_a'' \rho_b' \rho_c' = 1, \quad \rho_a' \rho_b'' \rho_c' = 1, \quad \rho_a' \rho_b' \rho_c'' = 1.$

No holes are ordinary. Every family under this type has a formula

$$\begin{aligned} y_a' &= y_b' = y_c', \\ y_a'' &= y_b'' = y_c''. \end{aligned}$$

This type has its classes characterized by the invariant

$$\kappa = \frac{C_a}{C_b},$$

or, to put it as an equation in homogeneous form, the invariant equation

$$\kappa' C_a - \kappa'' C_b = 0, \tag{21}$$

where κ' and κ'' are never both zero. Each value of $\frac{\kappa''}{\kappa'}\left(\text{or } \frac{\kappa'}{\kappa''}\right)$ gives a class, if we except the possibility of C_a and C_b both vanishing, which gives an additional class. There are an infinite number of classes under this type.

Class 1. $\kappa'' = 0$. a is semisingular, b and c logarithmic. In this class, as well as in all other classes under this type except Class 4, there is but one branch, and its constant multiples, everywhere fundamental.

Class 2. $\kappa' = 0$. b is semisingular, a and c logarithmic.

Class 3. $\kappa' + \kappa'' = 0$. c is semisingular, a and b logarithmic.

Class 4. (21) is identically satisfied. All holes are semisingular. All members are everywhere fundamental.

The remaining classes are infinite in number, each corresponding to a value of $\frac{\kappa''}{\kappa'}$ other than $0, \infty, -1$. In all of them all holes are logarithmic.

In conclusion, we deduce from this table the following theorems:

I. *The condition that some product* $\rho_a \rho_b \rho_c = 1$ *is sufficient, as well as necessary, that a family have an everywhere fundamental branch.*

II. *A necessary, but obviously not a sufficient condition that a hole be semisingular is that at least four products* $\rho_a \rho_b \rho_c = 1$. Theorem I shows that a family which has a semisingular hole must have an everywhere fundamental branch.

III. *The classes of Types II and III are completely characterized by the multipliers belonging to their everywhere fundamental branches.* With Type IV this is no longer true.

§ 5. THE FOUR EQUATIONS PROCEEDING FROM THE RELATION $S_c S_b S_a = 1$.

Let $S_c S_b S_a = \begin{pmatrix} m_1 & n_1 \\ m_2 & n_2 \end{pmatrix}$. Then the equation $S_c S_b S_a = 1$ is readily seen to be in reality equivalent to the four

$$m_1 = 1, \quad n_1 = 0, \quad m_2 = 0, \quad n_2 = 1.$$

The constants m_1, n_1, m_2, n_2, may be obtained directly in terms of the coefficients of a connecting formula, the associated constants C, and the multipliers, by performing the substitutions indicated. We are thus led to four equations connecting these latter quantities, which will be found equivalent to three such equations besides the relation $\rho_a' \rho_a'' \rho_b' \rho_b'' \rho_c' \rho_c'' = 1$. Other methods more convenient than using the substitution product $S_c S_b S_a$ suggest themselves; for instance, to use the equation in the form $S_b S_a = S_c^{-1}$. Without going into the details of this computation, we here append, under four cases, the system of equations sought for, giving them in as simple and symmetrical form as possible. With the first set it will be interesting to compare the equations given by Riemann for the class of hypergeometric families which he considers.[1]

1. $\rho_a' \neq \rho_a'', \ \rho_b' \neq \rho_b'', \ \rho_c' \neq \rho_c''.$

$$(\rho_a' \rho_b' \rho_c' - 1)\, \alpha_1 \delta = (\rho_a'' \rho_b' \rho_c' - 1)\, \beta \gamma_1,$$
$$(\rho_a' \rho_b'' \rho_c' - 1)\, \alpha_1 \gamma = (\rho_a'' \rho_b'' \rho_c' - 1)\, \alpha \gamma_1,$$
$$(\rho_a' \rho_b' \rho_c'' - 1)\, \beta_1 \delta = (\rho_a'' \rho_b' \rho_c'' - 1)\, \beta \delta_1,$$
$$(\rho_a' \rho_b'' \rho_c'' - 1)\, \beta_1 \gamma = (\rho_a'' \rho_b'' \rho_c'' - 1)\, \alpha \delta_1.$$

2. $\rho_a' = \rho_a'', \ \rho_b' \neq \rho_b'', \ \rho_c' \neq \rho_c''.$

$$(\rho_a' \rho_b' \rho_c' - 1)(\alpha_1 \delta - \beta \gamma_1) = C_a \alpha_1 \beta \rho_a' \rho_b' \rho_c',$$
$$(\rho_a' \rho_b'' \rho_c' - 1)(\alpha_1 \gamma - \alpha \gamma_1) = C_a \alpha_1 \alpha \rho_a' \rho_b'' \rho_c',$$
$$(\rho_a' \rho_b' \rho_c'' - 1)(\beta_1 \delta - \beta \delta_1) = C_a \beta_1 \beta \rho_a' \rho_b' \rho_c'',$$
$$(\rho_a' \rho_b'' \rho_c'' - 1)(\beta_1 \gamma - \alpha \delta_1) = C_a \alpha \beta_1 \rho_a' \rho_b'' \rho_c''.$$

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[1] Werke, p. 73.

3. $\rho_a' = \rho_a'', \rho_b' \neq \rho_b'', \rho_c' = \rho_c''.$

$$(\rho_a' \rho_b' \rho_c' - 1)(\alpha_1 \delta - \beta \gamma_1) = \rho_a' \rho_b' \rho_c' \, [C_c (\beta \delta_1 - \beta_1 \delta) + \beta \, C_a (\alpha_1 + C_c \beta_1)],$$
$$(\rho_a' \rho_b'' \rho_c' - 1)(\alpha_1 \gamma - \alpha \gamma_1) = \rho_a' \rho_b'' \rho_c' \, [C_c (\alpha \delta_1 - \beta_1 \gamma) + \alpha \, C_a (\alpha_1 + C_c \beta_1)],$$
$$(\rho_a' \rho_b' \rho_c' - 1)(\beta_1 \delta - \beta \delta_1) = \rho_a' \rho_b' \rho_c' \, C_a \beta_1 \beta,$$
$$(\rho_a' \rho_b'' \rho_c' - 1)(\beta_1 \gamma - \alpha \delta_1) = \rho_a' \rho_b'' \rho_c' \, C_a \alpha \beta_1.$$

4. $\rho_a' = \rho_a'', \rho_b' = \rho_b'', \rho_c' = \rho_c''.$

$$(\rho_a' \rho_b' \rho_c' - 1)(\alpha_1 \delta - \beta \gamma_1) = \rho_a' \rho_b' \rho_c' \, [C_c (\beta \delta_1 - \beta_1 \delta) + \beta \, C_a (\alpha_1 + C_c \beta_1)],$$
$$(\rho_a' \rho_b' \rho_c' - 1)(\alpha_1 \gamma - \alpha \gamma_1) = \rho_a' \rho_b' \rho_c' \, [C_c (\alpha \delta_1 - \beta_1 \gamma) + \alpha \, C_a (\alpha_1 + C_c \beta_1)] + C_b (\alpha_1 \delta - \beta \gamma_1),$$
$$(\rho_a' \rho_b' \rho_c' - 1)(\beta_1 \delta - \beta \delta_1) = \rho_a' \rho_b' \rho_c' \, C_a \beta_1 \beta,$$
$$(\rho_a' \rho_b' \rho_c' - 1)(\beta_1 \gamma - \alpha \delta_1) = \rho_a' \rho_b' \rho_c' \, C_a \alpha \beta_1 + C_b (\beta_1 \delta - \beta \delta_1).$$

PART II.

HYPERGEOMETRIC FAMILIES.

A. DEFINING PROPERTIES.

§ 1. REGULAR SINGULAR POINTS FOR BINARY FAMILIES. EXPONENTS.

In Part I, A, § 4, we have shown that if the hole a is a point, a basis (y_a', y_a'') has, in a properly chosen circle about a, the Laurent's developments

$$(22)\qquad \begin{aligned} y_a' &= (x-a)^{\alpha'}\sum_{\nu=-\infty}^{\nu=+\infty} g_\nu'(x-a)^\nu, \\ y_a'' &= (x-a)^{\alpha''}\sum_{\nu=-\infty}^{\nu=+\infty} g_\nu''(x-a)^\nu + \frac{C_a}{2\pi i}\, y_a' \log(x-a), \end{aligned}$$

provided a is finite; if a is the point infinity, we need only substitute $\frac{1}{x}$ for $(x-a)$ in (22), and this remark will apply throughout the subsequent work where we explicitly consider only the case where a is finite.

In the above-mentioned section, we have called a a *regular singular point* when the developments in (22) contain only a finite number of terms with negative values of ν; this is readily seen to be equivalent to the definition: *A finite regular singular point* a *of a binary family is an isolated singular point such that, if* (y', y'') *is some basis of the family, there exists a constant* α *for which the equations*

$$\begin{aligned} \lim_{x=a}\,(x-a)^\alpha y' &= 0, \\ \lim_{x=a}\,(x-a)^\alpha y'' &= 0, \end{aligned}$$

are always satisfied. These equations will then be true for *every* basis. We now examine in detail the fundamental bases of binary families for a regular singular point a.

If a is ordinary, a fundamental basis for that point always has the developments

$$(23)\qquad \begin{aligned} y_a' &= (x-a)^{\kappa'}\sum_{\nu=0}^{\nu=+\infty} g_\nu'(x-a)^\nu, \quad g_0' \neq 0, \\ y_a'' &= (x-a)^{\kappa''}\sum_{\nu=0}^{\nu=+\infty} g_\nu''(x-a)^\nu, \quad g_0'' \neq 0. \end{aligned}$$

Since $\rho_a' = e^{2\pi\iota\kappa'}$, $\rho_a'' = e^{2\pi\iota\kappa''}$, the difference $\kappa' - \kappa''$ cannot be an integer or zero, from our definition of the term ordinary.

If a is semisingular, formula (23) still holds for a fundamental basis (y_a', y_a''). Here we have $\kappa' - \kappa'' =$ an integer or zero, but evidently, since every basis is here fundamental for a, we can choose a basis for which $\kappa' \neq \kappa''$.

If a is logarithmic we have

$$y_a' = (x-a)^{\kappa'} \sum_{\nu=0}^{\nu=+\infty} g_\nu' (x-a)^\nu, \qquad g_0' \neq 0. \tag{24}$$

$$y_a'' = (x-a)^{\kappa''} \sum_{\nu=0}^{\nu=+\infty} g_\nu'' (x-a)^\nu + \frac{C_a}{2\pi i}\, y_a' \log(x-a).$$

Here $\kappa' - \kappa'' =$ an integer or zero, but we must note the possibility of all the coefficients g_ν'' vanishing, in which case κ'' has no meaning. Provided there is some $g_\nu'' \neq 0$ we suppose the notation so chosen in (24) that $g_0'' \neq 0$.

In all these cases we may observe a property of the branches in a fundamental basis which is most important, namely that expressed by the equation

$$\lim_{x=a} (x-a)^{-\lambda} [\log(x-a)]^{-k} y = g_0, \quad g_0 \neq 0,$$

a notation which covers all cases, if k is either 0 or 1. When such an equation is satisfied, y is said to have the exponent λ at a.[1]

In the ordinary case, y_a' and y_a'' have the exponents κ' and κ'' respectively at a. These we rename λ' and λ''. For a semisingular point a, y_a' and y_a'' again have the exponents κ' and κ''. When y_a' and y_a'' are so chosen that $\kappa' - \kappa'' > 0$, we rename these exponents λ' and λ'', respectively. When a is logarithmic, the exponent of y_a' at a is κ'; if $\kappa' - \kappa'' > 0$, the exponent of y_a'' is κ'', if $\kappa' - \kappa'' \leqq 0$, the exponent of y_a'' is κ'. Again we rename the exponents λ' and λ'', λ' being the exponent of y_a' and λ'' the exponent of y_a''. In all cases the general member of the family, $c_1 y_a' + c_2 y_a''$, can have for its exponent at a but one of the two quantities λ' and λ''. It will be natural, then, to speak of λ' and λ'' as the *exponents of the family at* a. Let us remember that

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[1] We may consider this definition as a particular case of the following: A function $\phi(x)$ is said to have the exponent λ at the point a, if there exists some positive integer k (including $k = 0$), such that no matter how x approaches a, being confined, however, to the neighborhood of a, we always have

$$\lim_{x=a} (x-a)^{-\lambda} [\log(x-a)]^{-k} \phi(x) = M,$$

where M is some constant $\neq 0$. This seemingly arbitrary definition is nevertheless one which applies to all solutions of homogeneous linear differential equations of any order at a regular point. Students of that subject will readily satisfy themselves that λ is one of the *exponents* of a in the sense in which that term is used for such differential equations.

when a is semisingular, our notation is so chosen that $\lambda' - \lambda'' > 0$, while, for a logarithmic, $\lambda' - \lambda'' \geqq 0$. Equal exponents, then, occur only when a is logarithmic.

A new notation will be found useful to indicate a basis fundamental at a with exponents λ', λ''; such a basis we designate by $(y^{(\lambda')}, y^{(\lambda'')})$. This coincides with the notation (y_a', y_a'') except in the case of a semisingular point. We have, then, the formula applicable to all cases

$$\begin{aligned} y^{(\lambda')} &= (x-a)^{\lambda'} \sum_{\nu=0}^{\nu=+\infty} g_\nu' (x-a)^\nu, \\ y^{(\lambda'')} &= (x-a)^{\lambda''} \sum_{\nu=0}^{\nu=+\infty} g_\nu'' (x-a)^\nu + \frac{C_a}{2\pi i}\, y^{(\lambda')} \log(x-a), \end{aligned} \tag{25}$$

where $g_0' \neq 0$; $g_0'' \neq 0$ when $\lambda' \neq \lambda''$, but we may have $g_0'' = 0$ when $\lambda' = \lambda''$, in which case a is logarithmic and $C_a \neq 0$.

It is also true that a binary family has two exponents at every point in T_n; the family being analytic at such points, these exponents must be positive integers.

The cases where the holes for a family are regular singular points, one or two in number, are of comparatively little interest, for the functions there involved are of a very elementary form. We shall be concerned hereafter with the special class of binary families whose holes are all regular singular points, *three* in number. Such families we call *Hypergeometric Families*.[1] At the three regular singular points a, b, c, such a family will have exponents λ', λ''; μ', μ''; ν', ν'', respectively, and formulæ analogous to (25) will exist for each point.

§ 2. THE EXPONENT SUM AND APPARENTLY SINGULAR POINTS. THE DETERMINANT *D*.

The exponents are a first essential in the definition of a hypergeometric family; they are, however, not entirely independent of each other, since they are subject to the relation

$$\lambda' + \lambda'' + \mu' + \mu'' + \nu' + \nu'' = \text{an integer}.$$

The positions of the regular singular points a, b, c are, from one point of view, nonessential, since they can be carried over by a linear transformation into any other three without changing the exponents, but we must remember that we have then a new independent variable. In particular, a, b, c can be carried over into 0, ∞, 1 respectively, by the transformation

$$x' = \frac{(x-a)(b-c)}{(x-b)(a-c)}. \tag{26}$$

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[1] *Cf.* Klein's term "Hypergeometrische Functionen," which he ultimately extends to all cases here considered. We shall find (in C, § 2) that all members of such families are expressible in terms of elementary functions and hypergeometric series.

We have seen that the exponent sum must be an integer; we may go further and state that this integer must be 1, 0, or negative. This result is obtained from a study of the functional determinant

$$D = \begin{vmatrix} y' & \dfrac{d\,y'}{d\,x} \\ y'' & \dfrac{d\,y''}{d\,x} \end{vmatrix}$$

If we compute $\dfrac{d\,y^{(\lambda')}}{d\,x}$ and $\dfrac{d\,y^{(\lambda'')}}{d\,x}$ from (25), we have

$$\begin{aligned} \frac{d\,y^{(\lambda')}}{d\,x} &= (x-a)^{\lambda'-1}\,[g_0{}'\,\lambda' + g_1{}'\,(\lambda'+1)\,(x-a) + \ldots], \\ \frac{d\,y^{(\lambda'')}}{d\,x} &= (x-a)^{\lambda''-1}\,[g_0{}''\,\lambda'' + g_1{}''\,(\lambda''+1)\,(x-a) + \ldots] \\ &\qquad + \frac{C_a}{2\,\pi\,i}\left[(x-a)^{\lambda'-1}\sum_{\nu=0}^{\nu=+\infty} g_\nu{}'\,(x-a)^\nu + \frac{d\,y^{(\lambda)}}{d\,x}\log\,(x-a)\right]. \end{aligned}$$

As we have already remarked, p. 17, a binary function and its derivative are always kindred. Accordingly we have, using the relation of page 15,

$$D = \operatorname{Det} A \cdot \begin{vmatrix} y^{(\lambda')} & \dfrac{d\,y^{(\lambda')}}{d\,x} \\ y^{(\lambda'')} & \dfrac{d\,y^{(\lambda'')}}{d\,x} \end{vmatrix} = \operatorname{Det} A \cdot \begin{bmatrix} (x-a)^{\lambda'+\lambda''-1}\,\{g_0{}'\,g_0{}''\,(\lambda''-\lambda') + g_1\,(x-a) + \ldots\} \\ + \dfrac{C_a}{2\,\pi\,i}\,(x-a)^{2\lambda'-1}\,\{g_0{}'^2 + \bar{g}_1\,(x-a) + \ldots\} \end{bmatrix}.$$

Hence D can have no logarithmic terms. If a is not logarithmic, $C_a = 0$ and $g_0{}'\,g_0{}''\,(\lambda''-\lambda') \neq 0$, so that we have the development about a[1]

$$D = (x-a)^{\lambda'+\lambda''-1}\sum_{\nu=0}^{\nu=+\infty} g_\nu\,(x-a)^\nu, \quad g_0 \neq 0. \tag{27}$$

If a is logarithmic, and $\lambda' \neq \lambda''$, the same formula holds, since $\lambda' - \lambda''$ is an integer. If $\lambda' = \lambda''$, then $g_0{}'\,g_0{}''\,(\lambda''-\lambda') = 0$, but $g_0{}'^2\,C_a/2\,\pi\,i \neq 0$, while $2\,\lambda' - 1 = \lambda' + \lambda'' - 1$, so that (27) still holds. In all cases, therefore, (27) gives the development of D about a when a is finite. If a is the point infinity, the reader may easily verify the development

$$D = \left(\frac{1}{x}\right)^{\lambda'+\lambda''+1}\sum_{\nu=0}^{\nu=+\infty} g_\nu \left(\frac{1}{x}\right)^\nu, \quad g_0 \neq 0. \tag{28}$$

In the preceding paragraph we have discussed only the case of a regular singular point, but obviously analogous developments must hold at non-singular points; in fact if t denotes such a point, with exponents τ', τ'' (which must be positive integers or zero, and unequal), we have about t the developments analogous to (27)

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[1] See Klein, p. 222 *et seq.*, where only ordinary singular points are considered.

$$D = (x - t)^{\tau'+\tau''-1} \sum_{\nu=0}^{\nu=+\infty} g_\nu (x - t)^\nu, \quad g_0 \neq 0. \tag{29}$$

If t is the point infinity, we have about it a development analogous to (28).

Let us now take our regular singular points at 0, ∞, 1, as we may always do by means of transformation (26). We have the equation

$$D = x^{\lambda'+\lambda''-1} (x - 1)^{\nu'+\nu''-1} \phi(x), \tag{30}$$

where the function $\phi(x)$ has obviously no singularities in the finite region of the plane, since the only singular points of D are 0, ∞, 1, at which developments (27) and (28) hold. At the point ∞, $\phi(x)$ has the exponent of D in that point increased by $(\lambda' + \lambda'' - 1) + (\nu' + \nu'' - 1)$; *i. e.* the exponent

$$(\lambda' + \lambda'' - 1) + (\mu' + \mu'' + 1) + (\nu' + \nu'' - 1).$$

Since the exponent sum $\lambda' + \lambda'' + \mu' + \mu'' + \nu' + \nu''$ is always an integer, we see that $\phi(x)$ can have at most a pole in the point infinity. Hence $\phi(x)$ must be a polynomial[1] and its degree is obviously the negative of its exponent at infinity; *i. e.*, it is

$$1 - (\lambda' + \lambda'' + \mu' + \mu'' + \nu' + \nu'').$$

This number, therefore, must be a positive integer or zero. Hence *the exponent sum can have only the following values :*

$$+1, 0, -1, -2, \ldots$$

In particular ϕ *is a constant when, and only when, the exponent sum is* $+1$. (*Cf.* Klein, p. 224.)

Families whose exponent sum is $+1$ we shall call P *families,* using the notation which Riemann adopted in his celebrated paper, in which, however, his discussion is limited to families for which a, b, c are all ordinary. A family whose exponent sum is $1 - n$, we will call a Q *family* of order n, Q being again a notation used by Riemann.[2] Functions belonging to these families we designate P *functions* and Q *functions* respectively. We may, if we please, regard P families as special Q families, namely those of order zero.

The distinction between P and Q families can be given in another way: *No branch of a* P *family vanishes to an order higher than the first in any point other than* 0,

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[1] We should observe here that $\phi(x) \equiv 0$ is impossible, since this would make D vanish identically, — an absurdity, for y' and y'' are linearly independent. See § 4 of the first-mentioned paper of Riemann.

[2] In the paper: Ueber die Fläche vom kleinsten Inhalt, etc., for a family of order 2. Werke, p. 323.

∞, 1, *when these are the regular singular points.* In other words, every non-singular point of a P family has the exponents 0, 1. *Every* Q *family* (*of order* > 0) *has at least one branch which vanishes to an order higher than the first in some non-singular point.* This we may show by an examination of the polynomial $\phi(x)$, which enters in (30). We have

$$\phi(x) = k(x - s_1)^{\sigma_1}(x - s_2)^{\sigma_2} \dots (x - s_\kappa)^{\sigma_\kappa}, \tag{31}$$

where k is a constant $\neq 0$, and each of the exponents σ is a positive integer, their sum, the degree of $\phi(x)$, being n for a family of order n. The points $s_1, s_2, \dots, s_\kappa$ are here all different from $0, \infty, 1$; each is a vanishing point of D, and in fact these are the only such points other than $0, \infty, 1$. We will call them *apparently singular points*[1] (Poincaré); in particular, referring to (31), we shall call s_i a σ_i-fold apparently singular point. From (29) we see that t is an apparently singular point when, and only when, $\tau' + \tau'' > 1$, *i. e.* when at least one exponent of t is greater than 1. Hence, *an apparently singular point is one at which some branch of the family vanishes to an order higher than* 1. Since $\phi(x)$ is a constant for P families, they can have no apparently singular points.

Let us designate the exponents of s_i $(i = 1, 2, \dots, \kappa)$ by σ_i', σ_i''. Referring to (29) we may then say: *To a* σ_i*-fold apparently singular point belong two positive integral exponents* σ_i' *and* σ_i'' *whose sum is* $\sigma_i + 1$.

Each point s_i introduces a new parameter into the family; it will turn out that only the groups of P families are in all cases completely determined by the exponents and the regular singular points.[2]

It will be useful to introduce here a notation analogous to that used by Riemann for P functions. Accordingly we will designate by the symbol

$$Q\left(\begin{array}{cccc} a & b & c & \\ \lambda' & \mu' & \nu' & \\ \lambda'' & \mu'' & \nu'' & x \end{array}\right), \tag{32}$$

any Q family with the regular singular points and exponents indicated. If we wish to specialize still further by indicating the apparently singular points and their exponents, we will use the symbol

$$Q\left(\begin{array}{ccc||cccc} a & b & c & s_1 & s_2 & \dots & s_\kappa \\ \lambda' & \mu' & \nu' & \sigma_1' & \sigma_2' & \dots & \sigma_\kappa' \\ \lambda'' & \mu'' & \nu'' & \sigma_1'' & \sigma_2'' & \dots & \sigma_\kappa'' \; x \end{array}\right). \tag{33}$$

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[1] Cf. Klein's term "Nebenpunkte," p. 225.

[2] It is to be remembered here that to determine the *concrete* group, we must know the regular singular points. To be sure we may carry these into any other three by a suitable linear transformation, but this gives us a new independent variable, and we can make no direct comparisons with the groups of functions of the old variable. In order that we may compare groups of families, the families must have the same regular singular points when expressed as functions of the same variable. That this is the case we assume throughout this paper.

As it is merely a matter of notation, we can interchange arbitrarily amongst each other in this symbol the columns to the right of the vertical bars, and those to the left. To the right, the exponents σ_i', σ_i'' are interchangeable; to the left, a similar remark holds when a, b, c are ordinary, but is no longer true for a semisingular or logarithmic point (except, of course, in the case of equal exponents), since, by the notation already adopted in such a case, the upper exponent in the symbol must be the larger.

Let us note here an extension of a property observed by Riemann for P functions: If Q is a family with the symbol (32), then

$$\left(\frac{x-a}{x-b}\right)^{\delta} Q$$

is a Q family with the symbol

$$\bar{Q}\left(\begin{array}{cccc} a & b & c & \\ \lambda'+\delta & \mu'-\delta & \nu' & \\ \lambda''+\delta & \mu''-\delta & \nu'' & x \end{array}\right).$$

Further, $\bar{Q}$ will evidently have the same apparently singular points with the same exponents as Q. But a Q family is by no means completely defined, in general, even by (33), so that we must still calculate the relations between the remaining constants required to define Q and $\bar{Q}$; this we will carry out at the end of the next section for a very general class of Q families. In particular we have the relation

$$x^{\delta}(x-1)^{\epsilon}\, Q\left(\begin{array}{cccc} 0 & \infty & 1 & \\ \lambda' & \mu' & \nu' & \\ \lambda'' & \mu'' & \nu'' & x \end{array}\right) = \bar{Q}\left(\begin{array}{cccc} 0 & \infty & 1 & \\ \lambda'+\delta & \mu'-\delta-\epsilon & \nu'+\epsilon & \\ \lambda''+\delta & \mu''-\delta-\epsilon & \nu''+\epsilon & x \end{array}\right), \tag{34}$$

δ and ϵ being any numbers we please; and $\bar{Q}$ will have the same apparently singular points with the same exponents as Q.

We need not here repeat this work for the apparently singular points, but one case is of especial interest. If both σ_i' and σ_i'' are different from zero, and $\sigma_i' > \sigma_i''$, we obtain, by dividing every member by $(x-s_i)^{\sigma_i''}$, from our Q family with its regular singular points at 0, ∞, 1, a new one in which the exponents of s_i are $\sigma_i' - \sigma_i''$, 0, nothing else being changed in its symbol (33) except the exponents of the point infinity. In our subsequent work, therefore, we shall discuss only families all of whose apparently singular points have zero for one exponent. A σ-fold apparently singular point has, for such a family, exponents $\sigma + 1$, 0, and may be regarded as σ coincident apparently singular points, each of which is simple, *i. e.* has exponents 2, 0. Using this convention, we may state, following Klein (*cf.* pp. 226, 227), the theorem: *The order,* n, *of a* Q *family is equal to the number of its simple apparently singular points.*

§ 3. THE DIFFERENTIAL EQUATION. ACCESSORY PARAMETERS.

To complete the data necessary to determine a Q family, we turn to the differential equation of the family. We have seen in Part I, *A*, § 7, that if (y', y'') is any basis of a binary family, every member of that family satisfies a differential equation of the form

$$\frac{d^2 y}{d x^2} + p \frac{d y}{d x} + q y = 0;$$

and from the work given in that section we easily obtain the equations

$$p = -\frac{\frac{d}{d x} D}{D} = -\frac{d}{d x} \log D,$$

$$q = \frac{D'}{D},$$

where D denotes the functional determinant

$$\begin{vmatrix} y' & \dfrac{d y'}{d x} \\ y'' & \dfrac{d y''}{d x} \end{vmatrix},$$

and D' the determinant

$$\begin{vmatrix} \dfrac{d y'}{d x} & \dfrac{d^2 y'}{d x^2} \\ \dfrac{d y''}{d x} & \dfrac{d^2 y''}{d x^2} \end{vmatrix}.$$

For a Q family of order n whose symbol is

$$35) \qquad Q\begin{pmatrix} 0 & \infty & 1 & \| & s_1 & s_2 & \cdots & s_\kappa & \\ \lambda' & \mu' & \nu' & \| & \sigma_1 + 1 & \sigma_2 + 1 & \cdots & \sigma_\kappa + 1 & x \\ \lambda'' & \mu'' & \nu'' & \| & 0 & 0 & \cdots & 0 & \end{pmatrix},$$

we have, using the notation of the preceding section,

$$(36) \qquad p = \frac{1 - \lambda' - \lambda''}{x} + \frac{1 - \nu' - \nu''}{x - 1} - \frac{\sigma_1}{x - s_1} - \frac{\sigma_2}{x - s_2} - \cdots - \frac{\sigma_\kappa}{x - s_\kappa}.$$

The calculation of q for such a family is, perhaps, best performed as follows:

Obviously the derivatives of the members of the family whose symbol is (35) themselves constitute a Q family[1] for which D' is the functional determinant. D'

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[1] We must note one exception, namely where one branch of the family whose symbol is (35) is a constant. Here the family $\frac{d Q}{d x}$ is not a binary family, but $q \equiv 0$, a result which agrees with our subsequent formulæ.

has, then, no logarithmic terms. Further, it is easy to see that the exponent of D' at 0 is equal to or greater than $\lambda' + \lambda'' - 3$; similarly, at ∞, its exponent is as large as $\mu' + \mu'' + 3$, and at 1, as large as $\nu' + \nu'' - 3$. By a course of reasoning analogous to that used for the determinant D, we conclude that D' is of the form

$$x^{\lambda'+\lambda''-3}(x-1)^{\nu'+\nu''-3}\phi'(x),$$

where $\phi'(x)$, which may have roots at 0 and 1, is a polynomial whose degree we denote by m; the exponent of D' at ∞ is therefore

$$-(\lambda' + \lambda'' - 3 + \nu' + \nu'' - 3 + m).$$

But we have seen that this exponent is as large as $\mu' + \mu'' + 3$, so that we have

$$m \leqq n + 2.$$

We have, therefore, for q the value

$$q = \frac{1}{x(x-1)}\left[\frac{\phi'(x)}{kx(x-1)(x-s_1)^{\sigma_1}(x-s_2)^{\sigma_2}\cdots(x-s_\kappa)^{\sigma_\kappa}}\right].$$

But, if $\sigma_i > 1$, we may easily show that $\phi'(x)$ has a factor $(x-s_i)^r$ where $r \geqq \sigma_i - 1$, by differentiating twice the branches

$$(x-s_i)^{\sigma_i+1}[1 + c_1'(x-s_i) + \cdots],$$

$$1 + c_1''(x-s_i) + \cdots,$$

and substituting the results in D'. Hence we may write

$$q = \frac{1}{x(x-1)}\left[\frac{\psi(x)}{x(x-1)(x-s_1)(x-s_2)\cdots(x-s_\kappa)}\right],$$

where $\psi(x)$ is a polynomial of degree $\leqq \kappa + 2$. This, again, may be put into the form

$$q = \frac{1}{x(x-1)}\left[\frac{M_1}{x} + M_2 + \frac{M_3}{x-1} + \frac{A_1}{x-s_1} + \frac{A_2}{x-s_2} + \cdots + \frac{A_\kappa}{x-s_\kappa}\right], \tag{37}$$

where M_1, M_2, M_3, A_1, A_2, . . ., A_κ are constant with respect to x, but are as yet undetermined functions of the exponents and apparently singular points.

We may at once find M_1 by substituting a branch $Q^{(\lambda')}$ in the differential equation of the family, thus obtaining a set of equations which must be satisfied. The first of these is $M_1 = -\lambda'\lambda''$. By similarly substituting $Q^{(\mu')}$ and $Q^{(\nu')}$, respectively, we obtain the equations

$$M_2 = \mu'\mu'', \quad M_3 = \nu'\nu''.$$

The differential equation of every Q family whose symbol is (35) is, therefore,

$$\frac{d^2 Q}{dx^2} + \left[\frac{1-\lambda'-\lambda''}{x} + \frac{1-\nu'-\nu''}{x-1} - \sum_{i=1}^{i=\kappa} \frac{\sigma_i}{x-s_i}\right]\frac{dQ}{dx} + \left[-\frac{\lambda'\lambda''}{x} + \mu'\mu'' + \frac{\nu'\nu''}{x-1} + \sum_{i=1}^{i=\kappa}\frac{A_i}{x-s_i}\right]\frac{Q}{x(x-1)} = 0, \tag{38}$$

where the *accessory parameters*[1] A_i must satisfy a system of κ equations which we obtain from the conditions that (38) be satisfied by the κ branches

$$1 + c_1^{(i)}(x-s_i) + c_2^{(i)}(x-s_i)^2 + \dots, \quad (i = 1, 2, \dots, \kappa).$$

These equations we will not here develop for the general case, but will content ourselves with deducing them for families all of whose apparently singular points are simple; *i.e.* where $\sigma_i = 1$ $(i = 1, 2, \dots, \kappa = n)$. The differential equation of such a family, its regular singular points being at $0, \infty, 1$, may be written

$$\frac{d^2 Q}{dx^2} + \left[\frac{1-\lambda'-\lambda''}{x} + \frac{1-\nu'-\nu''}{x-1} - \sum_{i=1}^{i=n} \frac{1}{x-s_i}\right]\frac{dQ}{dx} + \left[-\frac{\lambda'\lambda''}{x} + \mu'\mu'' + \frac{\nu'\nu''}{x-1} + \sum_{i=1}^{i=n}\frac{A_i}{x-s_i}\right]\frac{Q}{x(x-1)} = 0. \tag{39}$$

This is to be satisfied by a branch whose development about s_i is

$$1 + a_1(x-s_i) + a_2(x-s_i)^2 + \dots; \tag{40}$$

substituting this in (39) we obtain a system of equations to be satisfied by the coefficients of (40). Of these, however, only the following two impose any condition on the accessory parameters:

$$\frac{A_i}{s_i(s_i-1)} - a_1 = 0,$$

$$2a_2 - 2a_2 + \frac{a_1 A_i}{s_i(s_i-1)} - \frac{A_i}{s_i(s_i-1)}\left(\frac{1}{s_i} + \frac{1}{s_i-1}\right) + a_1\left[\frac{1-\lambda'-\lambda''}{s_i} + \frac{1-\nu'-\nu''}{s_i-1} - {\sum_{j=1}^{j=n}}' \frac{1}{s_i-s_j}\right] + \frac{1}{s_i(s_i-1)}\left[-\frac{\lambda'\lambda''}{s_i} + \mu'\mu'' + \frac{\nu'\nu''}{s_i-1} + {\sum_{j=1}^{j=n}}'\frac{A_j}{s_i-s_j}\right] = 0,$$

where the symbol ${\sum_{j=1}^{j=n}}'$ indicates a sum in which j is to assume all integral values from 1 to n inclusive, except the value i. These equations impose one condition on the accessory parameters A. Giving to i, successively, the values $1, 2, \dots, n$, we have the n equations

$$A_i^2 - A_i s_i(s_i-1)\left[\frac{\lambda'+\lambda''}{s_i} + \frac{\nu'+\nu''}{s_i-1} + {\sum_{j=1}^{j=n}}'\frac{1}{s_i-s_j}\right] + s_i(s_i-1)\left[-\frac{\lambda'\lambda''}{s_i} + \mu'\mu'' + \frac{\nu'\nu''}{s_i-1} + {\sum_{j=1}^{j=n}}'\frac{A}{s_i-s_j}\right] = 0. \tag{41}$$

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[1] A term used by Klein in his lectures since 1890.

When the accessory parameters are given a set of values satisfying (41), *the solutions of* (39) *constitute a* Q *family which is completely determined and whose symbol is*

$$(42) \qquad Q\left(\begin{array}{ccc|cccc} 0 & \infty & 1 & s_1 & s_2 & & s_n \\ \lambda' & \mu' & \nu' & 2 & 2 & \ldots & 2 \\ \lambda'' & \mu'' & \nu'' & 0 & 0 & \ldots & 0 \end{array} x\right).$$

Each set of solutions of (41) *defines a* Q *family with the symbol* (42). Hence P families only are in all cases completely determined by the exponents and the regular singular points.

A Q *family of order* n $\geqq 1$ *is completely determined when, and only when, we know in addition to its symbol* (35) *the values of its accessory parameters.*

We are now in a position to note the relation between the accessory parameters of the families Q and $\bar{Q}$ in (34) (see p. 36). This we may easily do by remembering that to obtain the differential equation of $\bar{Q}$ we have only to substitute in (38) for Q the expression

$$x^{-\delta}(x-1)^{-\epsilon}\,\bar{Q}.$$

If $A_1, A_2, \ldots, A_\kappa$ are the accessory parameters of Q, the accessory parameters $\bar{A}_1, \bar{A}_2, \ldots, \bar{A}_\kappa$ of $\bar{Q}$ are given by the equations

$$(43) \qquad \bar{A}_i = A_i + \sigma_i\,[\delta\,(s_i - 1) + \epsilon\, s_i] \quad (i = 1, 2, \ldots, \kappa).$$

We have given at some length the work of deducing the differential equation when the regular singular points are at $0, \infty, 1$; if they are at a, b, c, we may either use transformation (26) or proceed directly as in the previous case. We give here only the result; the equation is

$$(44) \qquad \frac{d^2 Q}{d x^2} + \left[\frac{1-\lambda'-\lambda''}{x-a} + \frac{1-\mu'-\mu''}{x-b} + \frac{1-\nu'-\nu''}{x-c} - \sum_{i=1}^{i=\kappa} \frac{\sigma_i}{x-s_i}\right]\frac{d Q}{d x}$$
$$+ \left[\frac{\lambda'\lambda''(a-b)(a-c)}{x-a} + \frac{\mu'\mu''(b-a)(b-c)}{x-b} + \frac{\nu'\nu''(c-a)(c-b)}{x-c} + \sum_{i=1}^{i=\kappa} \frac{B_i}{x-s_i}\right]\frac{Q}{(x-a)(x-b)(x-c)} = 0,$$

the accessory parameters being designated by B_i $(i = 1, 2, \ldots, \kappa)$.

In this section is the answer to Klein's[1] question as to the appearance of the differential equation of a Q family. Besides showing the complete data necessary to determine a Q family, the differential equations here developed afford an existence proof for families of arbitrary order n, with an exponent scheme arbitrarily given (subject to the condition that the exponent sum is $1 - n$). We shall also make use of this section in discussing the classification of Q families according to the scheme of Part I, B.

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[1] P. 234.

B. CRITERIA FOR THE CLASSES OF PART I, B.

§ 1. NECESSARY CONDITIONS FOR EACH CLASS IN TERMS OF THE EXPONENTS ALONE.[1]

In Part I, B, we first divided all possible binary families in a triply connected region into four types by means of characteristic relations amongst the multipliers. Remembering the connection between multipliers and exponents, we may write these criteria for Q families of any order, as follows:

Type I. $\lambda' + \mu' + \nu'$, $\lambda'' + \mu' + \nu'$, $\lambda' + \mu'' + \nu'$, $\lambda' + \mu' + \nu''$, are none of them integers.
" II. $\lambda' + \mu' + \nu'$ is an integer; $\lambda'' + \mu' + \nu'$, $\lambda' + \mu'' + \nu'$, $\lambda' + \mu' + \nu''$, are not integers.
" III. $\lambda' + \mu' + \nu'$, $\lambda'' + \mu' + \nu'$, are integers; $\lambda' + \mu'' + \nu'$, $\lambda' + \mu' + \nu''$, are not integers.
" IV. $\lambda' + \mu' + \nu'$, $\lambda'' + \mu' + \nu'$, $\lambda' + \mu'' + \nu'$, $\lambda' + \mu' + \nu''$, are all of them integers.

We hardly need repeat here what we have already noted for the more general case, that by a proper choice of notation every Q family will come under one, and but one, of these types. The conditions given are sufficient as well as necessary for each type.

We have seen in Part I, B, § 4, that in every class other than the one of Type I each family has an everywhere fundamental branch. Let us now examine the form of such a branch with exponents λ, μ, ν, corresponding to multipliers ρ_a, ρ_b, ρ_c, for a Q family of order n whose regular singular points we take at 0, ∞, 1. *Such a fundamental branch has, in fact, the form*

$$Q^{(\lambda)} = x^{\lambda} (x - 1)^{\nu} G(x), \tag{45}$$

where λ *stands for either* λ' *or* λ'', ν *for either* ν' *or* ν'', *and* $G(x)$ *is a polynomial.* For $G(x)$ is single valued, and is analytic, except in the point ∞, where it has the exponent $\lambda + \mu + \nu$, which must be an integer, since ∞ is not a branch point. Hence the only singularity of $G(x)$ is a possible pole at infinity, and $G(x)$ must therefore be a polynomial whose degree, the negative of its exponent at ∞, is $-(\lambda + \mu + \nu)$. We have, then, the necessary relation

$$\lambda + \mu + \nu = \text{either zero or a negative integer.}$$

From this result, combined with the table of Part I, B, § 4, which gives us the multipliers of the everywhere fundamental branches, and hence their exponents, we are enabled to give certain relations among the exponents of the regular singular points which must hold for the families of each class. We proceed to take these up in order.

Type I. There is but one class here, characterized in terms of the exponents above.

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[1] Compare Ritter's classification of P families in § 2 of his paper already cited. His numbering of classes differs from our scheme, though his types are the same. His criteria are easily reduced to ours.

Type II. $\lambda' + \mu' + \nu'$ an integer.

CLASS 1. The everywhere fundamental branch is

$$x^{\lambda'}(x-1)^{\nu'}\,G(x),$$

where $G(x)$, as in the remainder of this section, stands for some polynomial. Here $\lambda' + \mu' + \nu'$ is zero or a negative integer, its negative being the degree of $G(x)$.

CLASS 2. The everywhere fundamental branch is

$$x^{\lambda''}(x-1)^{\nu''}\,G(x).$$

$\lambda'' + \mu'' + \nu''$ is zero or a negative integer, its negative being the degree of $G(x)$. Since

$$\lambda' + \lambda'' + \mu' + \mu'' + \nu' + \nu'' = 1 - n,$$

we have here

$$\lambda' + \mu' + \nu' > -n.$$

CLASS 3. The two everywhere fundamental branches are

$$x^{\lambda'}(x-1)^{\nu'}\,G'(x),$$
$$x^{\lambda''}(x-1)^{\nu''}\,G''(x),$$

where the degrees of $G'(x)$, $G''(x)$ are $-(\lambda' \quad \mu' + \nu')$, $-(\lambda'' + \mu'' + \nu'')$, respectively. We have here

$$0 \geqq \lambda' + \mu' + \nu' < -n.$$

Type III. $\lambda' + \mu' + \nu'$, $\lambda'' + \mu' + \nu'$ are integers; $\lambda' - \lambda''$ is a positive integer or zero.

CLASS 1. The everywhere fundamental branch is

$$x^{\lambda'}(x-1)^{\nu'}\,G(x).$$

$G(x)$ is of degree $-(\lambda' + \mu' + \nu')$. We have here

$$\lambda' + \mu' + \nu' \leqq 0.$$

CLASS 2. The everywhere fundamental branch is

$$x^{\lambda'}(x-1)^{\nu''}\,G(x).$$

$G(x)$ is of degree $-(\lambda' + \mu'' + \nu'')$. We have

$$\lambda' + \mu'' + \nu'' \leqq 0,$$

which is equivalent to

$$\lambda' + \mu' + \nu' > \lambda' - \lambda'' - n.$$

Class 3. The two everywhere fundamental branches are

$$x^{\lambda''}(x-1)^{\nu'}G'(x),$$
$$x^{\lambda''}(x-1)^{\nu''}G''(x);$$

for this notation obviously includes the cases where the exponent of either branch at the point 0 is λ', when $G'(x)$, or $G''(x)$, has a factor $x^{\lambda'-\lambda''}$. A similar remark applies in the first four classes of Type IV. The degrees of $G'(x)$ and $G''(x)$ are $-(\lambda''+\mu'+\nu')$ and $-(\lambda''+\mu''+\nu'')$ respectively. We have here

$$\lambda''+\mu'+\nu' \leqq 0, \quad \lambda''+\mu''+\nu'' \leqq 0.$$

These two conditions are equivalent to

$$\lambda'-\lambda'' \geqq \lambda'+\mu'+\nu' > -n.$$

Type IV. All exponent sums $\lambda+\mu+\nu$ are integers. $\lambda'-\lambda''$, $\mu'-\mu''$, $\nu'-\nu''$ are all positive integers or zero, so that we have always

$$\lambda'+\mu'+\nu' > -\frac{n}{2}.$$

Class 1. The everywhere fundamental branch is

$$x^{\lambda''}(x-1)^{\nu'}G(x),$$

where $G(x)$ is of degree $-(\lambda''+\mu'+\nu')$. We have

$$\lambda''+\mu'+\nu' \leqq 0,$$

which is equivalent to

$$\lambda'-\lambda'' \geqq \lambda'+\mu'+\nu'.$$

This carries as a necessary consequence the relation

$$\lambda'+\mu'+\nu' > \mu'-\mu''+\nu'-\nu''-n.$$

Class 2. The everywhere fundamental branch is

$$x^{\lambda'}(x-1)^{\nu'}G(x),$$

where $G(x)$ is of degree $-(\lambda'+\mu''+\nu')$. Hence we have

$$\lambda'+\mu''+\nu' \leqq 0,$$

which is equivalent to

$$\mu'-\mu'' \geqq \lambda'+\mu'+\nu'.$$

This necessitates the relation

$$\lambda'+\mu'+\nu' > \lambda'-\lambda''+\nu'-\nu''-n.$$

CLASS 3. The everywhere fundamental branch is

$$x^{\lambda'}(x-1)^{\nu''}G(x),$$

where $G(x)$ is of degree $-(\lambda'+\mu'+\nu'')$. Hence we have

$$\lambda'+\mu'+\nu''\leqq 0,$$

which is equivalent to

$$\nu'-\nu''\geqq\lambda'+\mu'+\nu'.$$

This necessitates the relation

$$\lambda'+\mu'+\nu'>\lambda'-\lambda''+\mu'-\mu''-n.$$

CLASS 4. Every branch is here everywhere fundamental. We have

$$Q^{(\lambda')}=x^{\lambda'}(x-1)^{\nu''}G(x),$$

where $G(x)$ is of degree $\leqq-(\lambda'+\mu''+\nu'')$. Similarly considering $Q^{(\mu')}$ and $Q^{(\nu')}$, we have

$$\lambda'+\mu''+\nu''\leqq 0,\quad \lambda''+\mu'+\nu''\leqq 0,\quad \lambda''+\mu''+\nu'\leqq 0.$$

These conditions are equivalent to

$$\lambda'+\mu'+\nu'>\left\{\begin{array}{l}\lambda'-\lambda''-n,\\ \mu'-\mu''-n,\\ \nu'-\nu''-n.\end{array}\right.$$

REMAINING CLASSES. The everywhere fundamental branch is

$$x^{\lambda'}(x-1)^{\nu'}G(x),$$

where $G(x)$ is of degree $-(\lambda'+\mu'+\nu')$. Hence we have

$$\lambda'+\mu'+\nu'\leqq 0.$$

These necessary conditions we now give in the following condensed form:—

Type II. Class 1. $\lambda'+\mu'+\nu'\leqq 0.$
" 2. $\lambda'+\mu'+\nu'>-n.$
" 3. $0\geqq\lambda'+\mu'+\nu'>-n.$
Type III. Class 1. $\lambda'+\mu'+\nu'\leqq 0.$
" 2. $\lambda'+\mu'+\nu'>\lambda'-\lambda''-n.$
" 3. $\lambda'-\lambda''\geqq\lambda'+\mu'+\nu'>-n.$
Type IV. Class 1. $\lambda'-\lambda''\geqq\lambda'+\mu'+\nu'\ (>\mu'-\mu''+\nu'-\nu''-n).$
" 2. $\mu'-\mu''\geqq\lambda'+\mu'+\nu'\ (>\lambda'-\lambda''+\nu'-\nu''-n).$
" 3. $\nu'-\nu''\geqq\lambda'+\mu'+\nu'\ (>\lambda'-\lambda''+\mu'-\mu''-n).$
" 4. $\lambda'+\mu'+\nu'>\left\{\begin{array}{l}\lambda'-\lambda''-n,\\ \mu'-\mu''-n,\\ \nu'-\nu''-n.\end{array}\right.$

Remaining classes. $\lambda'+\mu'+\nu'\leqq 0.$

§ 2. SUFFICIENT AS WELL AS NECESSARY CONDITIONS.

In the case of P families (where $n = 0$), the necessary conditions of the preceding section are also sufficient that a family be of the designated class, but this is no longer true when $n > 0$. To be sure we can in many cases give sufficient conditions; thus, in Type II, the exponent sums determine the class of a family, if $\lambda' + \mu' + \nu'$ does not lie between 0 and $1 - n$ inclusive; in Type III, the same is true, if $\lambda' + \mu' + \nu'$ does not lie between $\lambda' - \lambda''$ and $1 - n$ inclusive; and we might also name cases where the class is determined by the exponent sums in Type IV. The cases where these means for determining the class of a family fail, we may conveniently refer to as the *doubtful cases*. In these doubtful cases we must refer to the differential equation of the family for sufficient conditions for the classes in terms of the constants which define a family, — *i. e.*, the exponents, the apparently singular points, and the accessory parameters.

It will be useful in this connection to obtain a general expression for a basis in Types II, III, IV. Let an everywhere fundamental branch (there must always be at least one for families of these types) be

$$Q' = x^{\lambda} (x - 1)^{\nu} G(x), \tag{46}$$

where $G(x)$ is of degree $-(\lambda + \mu + \nu)$. We have

$$D = \begin{vmatrix} Q' & \dfrac{dQ'}{dx} \\ Q'' & \dfrac{dQ''}{dx} \end{vmatrix} = x^{\lambda'+\lambda''-1} (x - 1)^{\nu'+\nu''-1} \phi(x),$$

where $\phi(x)$ has the value given by (31). This gives for Q'' the expression

$$\begin{aligned} Q'' &= Q' \left[\int \frac{D}{[Q']^2} \, dx + k_1 \right] \\ &= x^{\lambda} (x - 1)^{\nu} G(x) \left[\int x^{\lambda'+\lambda''-2\lambda-1} (x - 1)^{\nu'+\nu''-2\nu-1} \cdot \frac{\phi(x)}{[G(x)]^2} \, dx + k_1 \right]. \end{aligned} \tag{47}$$

Types II and III have their classes characterized by the everywhere fundamental branches which have been given in the preceding section for each class. We get a necessary and sufficient condition that a family have an everywhere fundamental branch (46), with exponents λ, μ, ν, by substituting (46), the coefficients of $G(x)$ being undetermined, in the differential equation (38) of the family. In addition to equations determining the coefficients of $G(x)$, we shall in this way also obtain a system of relations between the exponents of the regular and apparently singular points, these points themselves, and the accessory parameters, which will give necessary and sufficient conditions that (46) satisfy (38),—*i. e.* that the family have a branch (46). The necessary and sufficient condition that the family have a second everywhere fundamental branch not a constant multiple of the first can be obtained in the same way, or, having found the coefficients of $G(x)$, we may find this condition from the expression (47). We will satisfy ourselves here with merely indicating the plan of attack; in subdivision E we will carry the work through for families of order 1.

The above-indicated method fails in Type IV, since the classes here cannot always be distinguished by means of the everywhere fundamental branches, except Class 4, which is the only one with two such branches linearly independent. But in every class there is an everywhere fundamental branch

$$Q' = x^{\lambda''}(x-1)^{\nu''}\,G'(x).$$

Remembering that $\lambda' - \lambda''$, $\mu' - \mu''$, $\nu' - \nu''$ are all integers, we see that the integrand of (47) is here a rational function, so that we obtain Q'' by a mere quadrature of a simple kind. Since 0, ∞, 1 are the only regular singular points of our family, it is easy to see that we must have

$$(48)\qquad Q'' = x^{\lambda''}(x-1)^{\nu''}\,G''(x) + Q'\,[\kappa''\log x + \kappa'\log(x-1)],$$

where $G''(x)$ is of degree not greater than $-(\lambda'' + \mu'' + \nu'')$, and κ', κ'' are not functions of x. These numbers, κ' and κ'', can be used, as we have seen in Part I, B, § 4, to characterize by their ratio each class.[1] Substituting in (38) the form of (48) which characterizes each class, we shall come out with the conditions we desire.

C. RELATIONS BETWEEN KINDRED Q FUNCTIONS.

§ 1. GENERAL FORM OF RELATIONS.[2]

We have already noted, in that section, that the relations of Part I, A, § 7, take on a comparatively simple form for families having only regular singular points. We now proceed to examine these relations in detail for hypergeometric families; and we shall here not limit our considerations to families all of whose apparently singular points have zero for one exponent; our remarks apply equally to cases where the exponents of the apparently singular points are any pairs of (unequal) positive integers.

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[1] In fact we have so chosen our notation that $\frac{\kappa''}{\kappa'}$ has precisely the meaning here that it has on page 27 of Part I.

[2] See § 6 of Riemann's paper IV, for his treatment of this subject for the classes of P families there considered.

We shall find a slight complication of our results possible for families having a semisingular point; accordingly we will at first confine ourselves to families of the classes where a semisingular point is impossible. Here we have $Q^{(\lambda')} = Q_a'$, $Q^{(\lambda'')} = Q_a''$.

Let (Q', Q''), $(\bar{Q}', \bar{Q}'')$, denote any pair of corresponding bases of any two kindred Q families which belong to one of the classes above indicated. Let the two families have the notation schemes

$$Q\begin{pmatrix} \overset{0}{\lambda'} & \overset{\infty}{\mu'} & \overset{1}{\nu'} & \\ \lambda'' & \mu'' & \nu'' & x \end{pmatrix} \text{ and } \bar{Q}\begin{pmatrix} \overset{0}{\bar{\lambda}'} & \overset{\infty}{\bar{\mu}'} & \overset{1}{\bar{\nu}'} & \\ \bar{\lambda}'' \cdot & \bar{\mu}'' & \bar{\nu}'' & x \end{pmatrix},$$

and let Q be of order n, $\bar{Q}$ of order $\bar{n}$. Corresponding exponents of the two families (and these are to be the ones similarly situated in the two symbols) differ by integers, since corresponding multipliers must be equal (see Theorem III, Part I, A, § 6). If we designate by $\overline{\lambda + \bar{\lambda}}$, the smaller in real part of the two sums, $\lambda' + \bar{\lambda}''$ and $\bar{\lambda}' + \lambda''$, using a similar notation with the other pairs of exponents, we obtain, by a process so close in details to the work of page 16 of Part I that the result can easily be written at once by comparison with equation (9), the relation: —

$$\begin{vmatrix} Q' & \bar{Q}' \\ Q'' & \bar{Q}'' \end{vmatrix} = x^{\overline{\lambda+\bar{\lambda}}}(x-1)^{\overline{\nu+\bar{\nu}}}\phi(x), \tag{48'}$$

$\phi(x)$ being single valued and analytic everywhere except in the point ∞, where it has a pole of order

$$-(\overline{\lambda + \bar{\lambda}} + \overline{\mu + \bar{\mu}} + \overline{\nu + \bar{\nu}}), \tag{49}$$

and is therefore a polynomial of degree given by (49).

If we write, using the ordinary notation for the absolute value of a number,

$$\begin{aligned} \Delta\lambda &= |(\lambda' - \lambda'') - (\bar{\lambda}' - \bar{\lambda}'')|, \\ \Delta\mu &= |(\mu' - \mu'') - (\bar{\mu}' - \bar{\mu}'')|, \\ \Delta\nu &= |(\nu' - \nu'') - (\bar{\nu}' - \bar{\nu}'')|, \end{aligned}$$

we have the equations

$$\begin{aligned} -(\overline{\lambda + \bar{\lambda}}) &= \frac{\Delta\lambda}{2} - \frac{\lambda' + \lambda'' + \bar{\lambda}' + \bar{\lambda}''}{2}, \\ -(\overline{\mu + \bar{\mu}}) &= \frac{\Delta\mu}{2} - \frac{\mu' + \mu'' + \bar{\mu}' + \bar{\mu}''}{2}, \\ -(\overline{\nu + \bar{\nu}}) &= \frac{\Delta\nu}{2} - \frac{\nu' + \nu'' + \bar{\nu}' + \bar{\nu}''}{2}, \end{aligned}$$

so that the degree of $\phi(x)$ is

$$\frac{\Delta\lambda + \Delta\mu + \Delta\nu + n + \bar{n}}{2} - 1. \tag{50}$$

Hence if Q, $\bar{Q}$, $\bar{\bar{Q}}$ *are any three kindred* Q *functions with regular singular points* 0, ∞, 1, *none of which are semisingular, relation* (10) *is, for these functions,*

$$Q\,\Phi(x) + \bar{Q}\,\bar{\Phi}(x) + \bar{\bar{Q}}\,\bar{\bar{\Phi}}(x) = 0, \tag{51}$$

where, after all common factors are divided out, $\Phi(x)$, $\bar{\Phi}(x)$, $\bar{\bar{\Phi}}(x)$, *are polynomials of degrees not higher than the numbers given by formulæ analogous to* (50).

In classes where semisingular points occur, every basis is fundamental at such a point, so that although two kindred families have the same connecting formulæ, *it is possible in some cases that they may have no common connecting formula in which the fundamental bases have exponents* λ', λ; μ', μ''; ν', ν''; *respectively.* The only new possibility here is that in such a case we must substitute in (48) and (49), if a is semisingular, $\lambda'' + \bar{\lambda}''$ for $\lambda + \bar{\lambda}$, and similarly if other points are semisingular. This merely increases the number given by (49) and (50) by an integer. The form of (51) is unchanged in that the coefficients are still polynomials, but the degrees of one or more of these polynomials may be larger than the limit (50) would give. But even this exception to the general rule can occur only in a comparatively limited range of cases under the classes where semisingular points enter, namely, where two of the three families to which the functions in (51) belong have no common connecting formula in which bases having exponents λ', λ'', μ', μ'', ν', ν'' enter.

§ 2. EXPRESSION OF Q FUNCTIONS IN TERMS OF KINDRED P FUNCTIONS.

We may apply the results of the preceding section in many ways. Thus, the differential equations (38) and (39) are special cases of relation (51). Again, Riemann, in his paper, "Ueber die Fläche vom kleinsten Inhalt, etc.,"[1] has given a formula connecting a special Q function of order 2 with a kindred P function and its derivative. But an especially important application is the expression of Q functions in terms of P functions and rational functions only. We shall find that no P family can be of Class 3 of Type I, and that the classes of Type IV, other than 1, 2, 3, 4, are likewise impossible for P families, but that in all other classes there are P families kindred to any Q families of those classes. From (51) we have the theorem: *A* Q *function whose family belongs to the classes possible for* P *families is expressible as a sum of terms each consisting of a kindred* P *function multiplied by a rational function of the independent variable.* (51) would, of course, give directly but two such terms, but each of these is expressible in terms of other P functions by means of similar formulæ.

We may note that in the cases where this representation fails, the families are composed of elementary functions. Thus in Type I, Class 3, the family is

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[1] Werke, p. 324.

$$c_1 x^{\lambda'}(x-1)^{\nu'} G'(x) + c_2 x^{\lambda''}(x-1)^{\nu''} G''(x);$$

in the classes of Type IV, other than 1, 2, 3, 4, the family is

$$c_1 x^{\lambda'}(x-1)^{\nu'} G'(x) + c_2 \{x^{\lambda''}(x-1)^{\nu''} G''(x) + x^{\lambda'}(x-1)^{\nu'} G'(x)\,[\kappa'' \log x + \kappa' \log (x-1)]\};$$

where, in both cases, $G'(x)$, $G''(x)$, are polynomials, while κ'' and κ' are non-vanishing constants.[1] It is a familiar fact that P functions are expressible in terms of elementary functions and hypergeometric series, so that we have established the result mentioned in the footnote to page 32: *All members of hypergeometric families are expressible in terms of elementary functions and hypergeometric series.*

The question naturally arises as to how simply a Q function can be expressed in terms of P functions. Thus Klein,[2] after showing that we can form a Q function of any order κ by taking a linear combination of suitable P functions, says: "Um eine Q Function in allgemeinster Weise zu bilden, bei der κ irgend einen vorgeschriebenen Wert hat, muss ich $\kappa - 1$ verwandte P Functionen . . . mit Hülfe constanter Coefficienten linear zusammensetzen." Whether this means that every Q function of order κ is so expressible is not clear. We must certainly except functions belonging to classes impossible for P families. From relation (50) we have no clue to this result; as we proceed step by step from a Q function to its expression in terms of the P functions indicated, the degrees of the coefficients would seem to increase, but common factors may, of course, occur. Looking at the question from another point of view, we have at our disposal the κ ratios of the undetermined coefficients of the expression Klein mentions, with which we may hope to form a Q function with the desired apparently singular points; but can we also so determine the Q function we have formed that it will have any of the possible sets of accessory parameters? This question the writer has not been able to answer, in general; in the case of Q functions of order 1, we shall see that, with the exceptions noted, the answer to the above question is affirmative.

D. *P* FAMILIES.

We are now in a position to develop the properties of P families as a corollary of the more general results of the preceding sections. Riemann, in his celebrated paper, confined himself to the case where the difference of no exponent pair is an integer; we shall here be able to extend his results to all cases.

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[1] In a footnote on pages 30 and 31 of Ritter's paper (Math. Ann., Vol. 48, 1896), Schilling has shown how we may pass from a Q function expressed in terms of P functions of Type I, Class 2, to a Q function of Class 3, all of order 1, by allowing coefficients to approach certain limits, and it seems likely that this process can be used in other cases.

[2] Page 229.

§ 1. CRITERIA FOR EACH CLASS OF *P* FAMILIES.

Riemann's definition of a P function includes Type II as well as part of Type I, but he seems to have overlooked the former in discussing the group. Miss Winston,[1] following out a suggestion of Klein's,[2] first supplied this deficiency by noting from the development in series of the fundamental branches the difference between families of Classes 1 and 2 of Type II. Ritter[3] and Schilling[4] have since obtained essentially the criteria which follow for all classes, the latter by geometrical methods, the former analytically, but starting from an artificial definition of kindred functions which he has not shown to be equivalent to the requirement that they have the same group. Our method has its chief claim to interest from the fact that it is a natural extension of Riemann's.

The criteria for the types are given on page 41. For the classes under them the necessary relations of page 44, when we put $n = 0$, become:

Type II. Class 1. $\lambda' + \mu' + \nu' \leqq 0$.
" 2. $\lambda' + \mu' + \nu' > 0$.
" 3. Impossible.
Type III. Class 1. $\lambda' + \mu' + \nu' \leqq 0$.
" 2. $\lambda' + \mu' + \nu' > \lambda' - \lambda''$.
" 3. $\lambda' - \lambda'' \geqq \lambda' + \mu' + \nu' > 0$.
Type IV. Class 1. $\lambda' - \lambda'' \geqq \lambda' + \mu' + \nu' \; (> \mu' - \mu'' + \nu' - \nu'')$.
" 2. $\mu' - \mu'' \geqq \lambda' + \mu' + \nu' \; (> \lambda' - \lambda'' + \nu' - \nu'')$.
" 3. $\nu' - \nu'' \geqq \lambda' + \mu' + \nu' \; (> \lambda' - \lambda'' + \mu' - \mu'')$.
" 4. $\lambda' + \mu' + \nu' > \left\{ \begin{array}{l} \lambda' - \lambda'', \\ \mu' - \mu'', \\ \nu' - \nu''. \end{array} \right.$

The necessary relation $\lambda' + \mu' + \nu' > -\frac{n}{2}$ makes it impossible, in Type IV, that $\lambda' + \mu' + \nu'$ be equal to or less than zero, so that the remaining classes are impossible.

But these relations include all possible values of $\lambda' + \mu' + \nu'$, and in each type they are mutually exclusive. Hence, *the above relations are, for each class of any type, a necessary and sufficient condition that a family of that type belong to that class.*

The point a is semisingular in Class 3 of Type III, and in Classes 1 and 4 of Type IV, and in no other classes. By combining the criteria for these classes, we obtain a necessary and sufficient condition that a be semisingular. This condition is: *either* $\lambda' + \mu' + \nu'$ *or* $\lambda' + \mu' + \nu''$ *is an integer of the sequence* $1, 2, \ldots, \lambda' - \lambda''$.[5]

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[1] Math. Ann., Vol. 46, p. 157, 1895.

[2] Pages 413–415.

[3] In § 2 of his paper. Math. Ann., Vol. 48, 1896.

[4] Math. Ann., Vol. 46, p. 538, 1895.

[5] Compare the conditions given in Schwarz's paper, Ueber diejenigen Fälle in welchen die Gaussische hypergeometrische Reihe eine algebraische Function ihres vierten Elementes darstellt. Crelle, Vol. 75, pp. 292–335, 1873.

Since our criteria, both for types and classes, are entirely in terms of the exponents, *P families with the same exponents at the same points are kindred.* So much we conclude from this section; in the next section we show that a P family is in fact determined by its exponents.

§ 2. THE *P* FAMILY DETERMINED BY ITS EXPONENTS.

From the differential equation of the P family, we could infer that its only parameters are the exponents, but it will be interesting to follow Riemann here. We wish to prove that but one P family, with given singular points, can have a given set of exponents.

We shall need here the following lemma, which is true for P families but not for all Q families:

If two P *families are kindred, they have a common connecting formula* (*including the associated constants* C) *in which the fundamental bases have exponents* λ', λ''; μ', μ''; ν', ν'', *respectively.*

This follows, as a matter of course, from Theorem III of Part I, A, § 6, when a, b, c are none of them semisingular. But when, for instance, a is semisingular, every basis is fundamental at a, and hence the two branches of a basis may both have exponents λ''. We proceed, therefore, to show the truth of our lemma for each class where semisingular points enter.

Type III, Class 3. We have, from Part I, B, § 4,

$$P_a' = \alpha P_b' = \alpha_1 P_c',$$
$$P_a'' = \delta P_b'' = \delta_1 P_c''.$$

The exponents of P_a' and P_a'' must both be λ'', since the fact that the sum of the three exponents of an everywhere fundamental branch must be less than or equal to zero is readily seen to bar out any other possibilities. Hence we have

$$P^{(\lambda')} = \alpha P^{(\mu')} + \beta P^{(\mu'')} = \alpha_1 P^{(\nu')} + \beta_1 P^{(\nu'')},$$
$$P^{(\lambda'')} = \delta P^{(\mu'')} = \delta_1 P^{(\nu'')}.$$

Here none of the coefficients can vanish, and in any two families of this class we can take α, β, δ the same; then if one family has the above formula, any kindred family must have the same formula, since a basis $(P^{(\lambda')}, P^{(\lambda'')})$ of the first family corresponds to a basis $(\bar{P}^{(\lambda')}, \bar{P}^{(\lambda'')})$ of the kindred family; for connecting formulæ involving these bases must exist which are the same for the two families; and to complete our proof we need only observe that P_b', P_b'' have exponents μ' and μ'', respectively, and P_c', P_c'', exponents ν', ν''.

Type IV, Class 1. We easily see that the connecting formulæ can always be taken

$$P^{(\lambda')} = \beta\, P^{(\mu'')} = \beta_1\, P^{(\nu'')},$$
$$P^{(\lambda'')} = \gamma\, P^{(\mu')} = \gamma_1\, P^{(\nu')}.$$

Having chosen $P^{(\mu')}$ and $P^{(\mu'')}$, so that C_b has any preassigned value $\neq 0$, we may, by our choice of $P^{(\lambda')}$, $P^{(\lambda'')}$, give β and γ any values we please $\neq 0$. We complete our proof here as for Class 3 of Type III. Classes 2 and 3 may be discussed in the same way.

Type IV, Class 4. We have

$$P_a' = a\, P_b' = a_1\, P_c',$$
$$P_a'' = \delta\, P_b'' = \delta_1\, P_c''.$$

Every basis is here fundamental at every hole. Taking P_a' and P_a'' as $P^{(\lambda')}$ and $P^{(\lambda'')}$, respectively, we may easily assure ourselves that the only possible exponents for P_b', P_c', are μ'', ν'', and that we have always a formula

$$P^{(\lambda')} = \beta\, P^{(\mu'')} = \beta_1\, P^{(\nu'')},$$
$$P^{(\lambda'')} = \gamma\, P^{(\mu')} = \gamma_1\, P^{(\nu')} + \delta_1\, P^{(\nu'')},$$

where none of the coefficients vanish, but are otherwise arbitrary.

The proof of our lemma being thus completed, we now proceed to the proof of the theorem:

Only one P *family with given singular points can have a given set of exponents.*

Take the given singular points at 0, ∞, 1 by means of transformation (26) and let the given set of exponents be λ', λ'', μ', μ'', ν', ν'' for a family P. Let $(P^{(\lambda')}, P^{(\lambda'')})$, $(P^{(\mu')}, P^{(\mu'')})$, $(P^{(\nu')}, P^{(\nu'')})$ be fundamental bases of this family, with the exponents indicated, and suppose we are given a connecting formula in which all of them enter. Then by our lemma, since families with the same exponents and singular points are always kindred (see preceding section), a second family $\bar{P}$ with the same exponents and singular points must have the *same* connecting formula in terms of corresponding bases $(\bar{P}^{(\lambda')}, \bar{P}^{(\lambda'')})$, $(\bar{P}^{(\mu')}, \bar{P}^{(\mu'')})$, $(\bar{P}^{(\nu')}, \bar{P}^{(\nu'')})$. This is the point on which Riemann's work rests; his proof may now be easily applied to our more general problem. In fact we have only to follow the work of Part I, A, § 7, to see that here, as well as in the case Riemann considers, the determinant

$$\begin{vmatrix} P^{(\lambda')} & \bar{P}^{(\lambda')} \\ P^{(\lambda'')} & \bar{P}^{(\lambda'')} \end{vmatrix}$$

is equal to the function

$$x^{\lambda'+\lambda''}\,(x-1)^{\nu'+\nu''}\,\bar{\phi}\,(x),$$

where $\bar{\phi}(x)$ is everywhere on the sphere a single valued analytic function, and hence is a constant. Further, since the exponent at ∞ of $\bar{\phi}(x)$ is

$$\lambda' + \lambda'' + \mu' + \mu'' + \nu' + \nu'' = 1,$$

$\bar{\phi}(x)$ vanishes at ∞, and therefore is identically zero. From this result, and the common connecting formula of the two families, we deduce, with Riemann, the equations

$$\frac{\bar{P}^{(\lambda')}}{P^{(\lambda')}} = \frac{\bar{P}^{(\lambda'')}}{P^{(\lambda'')}} = \frac{\bar{P}^{(\mu')}}{P^{(\mu')}} = \frac{\bar{P}^{(\nu')}}{P^{(\nu')}}. \tag{52}$$

Now $\frac{\bar{P}^{(\lambda')}}{P^{(\lambda')}}$ is clearly single valued in the neighborhood of the point 0, and similarly for $\frac{\bar{P}^{(\mu')}}{P^{(\mu')}}$, $\frac{\bar{P}^{(\nu')}}{P^{(\nu')}}$, in the neighborhood of ∞, 1, respectively. Hence from equations (52) we see that $\frac{\bar{P}^{(\lambda')}}{P^{(\lambda')}}$ is single valued and analytic on the entire sphere, provided that $P^{(\lambda')}$ and $P^{(\lambda'')}$ do not both vanish at any point other than $0, \infty, 1$. To prove that this is the case, we need only remember that the functional determinant

$$D = \begin{vmatrix} P^{(\lambda')} & \dfrac{d\,P^{(\lambda')}}{d\,x} \\ P^{(\lambda'')} & \dfrac{d\,P^{(\lambda'')}}{d\,x} \end{vmatrix}$$

is equal to $x^{\lambda'+\lambda''-1}\,(x-1)^{\nu'+\nu''-1}\,\phi(x)$ (see page 34), $\phi(x)$ being for P families a constant, and as observed in the footnote to page 34, $\phi(x) \not\equiv 0$. From this it is obvious that $P^{(\lambda')}$, $P^{(\lambda'')}$ cannot vanish together in any point other than $0, \infty, 1$.

It is evident, therefore, that $\frac{\bar{P}^{(\lambda')}}{P^{(\lambda')}} = \frac{\bar{P}^{(\lambda'')}}{P^{(\lambda'')}}$, being single valued and analytic over the entire sphere, must be a constant. In consequence, the families

$$c_1 P^{(\lambda')} + c_2 P^{(\lambda'')} \text{ and } \bar{c}_1 \bar{P}^{(\lambda')} + \bar{c}_2 \bar{P}^{(\lambda'')},$$

must be identical, and this establishes our theorem.

Note that this proof fails for Q families of order > 0; in fact the theorem itself is not true.

§ 3. RELATIONS BETWEEN KINDRED P FUNCTIONS.

We may take our formulæ here bodily from Part II, C, § 1. We have

$$\begin{vmatrix} P' & \bar{P}' \\ P'' & \bar{P}'' \end{vmatrix} = x^{\overline{\lambda+\bar{\lambda}}}\,(x-1)^{\overline{\nu+\bar{\nu}}}\,\phi(x),$$

and here $\phi(x)$ is always of degree

$$\frac{\Delta\lambda + \Delta\mu + \Delta\nu}{2} - 1,$$

on account of the lemma of the preceding section. Formula (51) we can use with merely a change of notation.

§ 4. THE DIFFERENTIAL EQUATION.

The differential equation of a P family is easily written down from the more general equations of Part II, A, § 3. Putting $n = 0$, we have from (39), if the regular singular points of our family are 0, ∞, 1,

$$\frac{d^2P}{dx^2} + \left[\frac{1-\lambda'-\lambda''}{x} + \frac{1-\nu'-\nu''}{x-1}\right]\frac{dP}{dx} + \left[-\frac{\lambda'\lambda''}{x} + \mu'\mu'' + \frac{\nu'\nu''}{x-1}\right]\frac{P}{x(x-1)} = 0;$$

and from (44) we obtain for the case where the singular points are a, b, c the equation first given by Papperitz,[1]

$$(53)\qquad \frac{d^2P}{dx^2} + \left[\frac{1-\lambda'-\lambda''}{x-a} + \frac{1-\mu'-\mu''}{x-b} + \frac{1-\nu'-\nu''}{x-c}\right]\frac{dP}{dx}$$

$$+ \left[\frac{\lambda'\lambda''(a-b)(a-c)}{x-a} + \frac{\mu'\mu''(b-a)(b-c)}{x-b} + \frac{\nu'\nu''(c-a)(c-b)}{x-c}\right]\frac{P}{(x-a)(x-b)(x-c)} = 0.$$

The differential equation (53) *affords an existence proof for a P family with any three singular points and any exponents whose sum is 1.* It therefore gives an existence proof for all the classes we have characterized in terms of the exponents on page 50.

E. Q FAMILIES OF ORDER 1.

§ 1. THE DIFFERENTIAL EQUATION. THE FAMILIES Q AND $\bar{Q}$.

In the first three sections of the paper of Ritter which we have so often cited, are given a discussion of the differential equation of the general Q family of order 1, applications to a classification of P families according to their groups, and an example of the expression of a Q function linearly in terms of P functions. For many points of interest the reader may refer to that paper; here we shall only treat the problem of classifying Q families of order 1, and add a few remarks about the expression of Q functions of order 1 in terms of P functions. For our purposes, it is sufficient to take the regular singular points at 0, ∞, 1. The apparently singular point s (which must be simple) is not to coincide with any of these.

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[1] Math. Ann., Vol. 25, 1885.

The differential equation may be read off directly from (39); it is

$$(54)\qquad \frac{d^2 Q}{d x^2} + \left[\frac{1-\lambda'-\lambda''}{x} + \frac{1-\nu'-\nu''}{x-1} - \frac{1}{x-s}\right]\frac{d Q}{d x} + \left[-\frac{\lambda'\lambda''}{x} + \mu'\mu'' + \frac{\nu'\nu''}{x-1} + \frac{A}{x-s}\right]\frac{Q}{x(x-1)} = 0,$$

and is thus given on page 6 of Ritter's paper. From (41) we see that A must be a solution of the equation,

$$(55)\qquad A^2 - A s(s-1)\left(\frac{\lambda'+\lambda''}{s} + \frac{\nu'+\nu''}{s-1}\right) + s(s-1)\left(-\frac{\lambda'\lambda''}{s} + \mu'\mu'' + \frac{\nu'\nu''}{s-1}\right) = 0.$$

If Q denotes a family which satisfies (54), having one of the solutions, A_1, of (55) for its accessory parameter, the results noted in sections 2 and 3 of Part II, A, show us that the family

$$(56)\qquad \bar{Q} = x^{\delta}(x-1)^{\epsilon} Q,$$

consists of the solutions of the differential equation

$$(57)\qquad \frac{d^2 \bar{Q}}{d x^2} + \left[\frac{1-\lambda'-\lambda''-2\delta}{x} + \frac{1-\nu'-\nu''-2\epsilon}{x-1} - \frac{1}{x-s}\right]\frac{d \bar{Q}}{d x}$$
$$+ \left[-\frac{(\lambda'+\delta)(\lambda''+\delta)}{x} + (\mu'-\delta-\epsilon)(\mu''-\delta-\epsilon) + \frac{(\nu'+\epsilon)(\nu''+\epsilon)}{x-1} + \frac{A_1+\delta(s-1)+\epsilon s}{x-s}\right]\frac{\bar{Q}}{x(x-1)} = 0.$$

We shall find this result of considerable use in the problem of characterizing our classification of Q families of order 1.

§ 2. CLASSIFICATION CRITERIA IN THE DOUBTFUL CASES.

In § 2 of Part II, B, we have seen that the relations in terms of the exponents alone, there given, completely determine the class of a family except in cases which we have termed *doubtful.* These we now investigate.

Type I. No doubtful case, as there is only one class.

Type II. The only doubtful cases are for families whose exponents satisfy the relation $\lambda' + \mu' + \nu' = 0$. In this case the fact that the exponent sum $\lambda' + \lambda'' + \mu' + \mu'' + \nu' + \nu''$ is zero gives us also $\lambda'' + \mu'' + \nu'' = 0$. For these cases we will deduce relations between the parameters of the family which we first show to be necessary for each class, and then to be sufficient.

The two roots of (55) will be found to be $-\lambda' - \mu' s$ and $-\lambda'' - \mu'' s$ in these doubtful cases.

Class 1. One member of the family must be $k x^{\lambda'}(x-1)^{\nu'}$, where k is a constant. (See page 42. The degree of $G(x)$ is $-(\lambda' + \mu' + \nu')$, which is here zero. Similarly refer to the work of pages 42–44 throughout the rest of this section.)

Hence if we put $\delta = -\lambda'$, $\epsilon = -\nu'$, a constant must satisfy (57). This gives the condition, necessary and sufficient, that $k\,x^{\lambda'}\,(x-1)^{\nu'}$ be a member of the family,

$$A = \lambda'(s-1) + \nu' s = -\lambda' - \mu' s. \tag{58}$$

CLASS 2. Proceeding similarly, we see that a necessary and sufficient condition that $k\,x^{\lambda''}\,(x-1)^{\nu''}$ be a member of the family is

$$A = \lambda''(s-1) + \nu'' s = -\lambda'' - \mu'' s. \tag{59}$$

CLASS 3. Here A must have both values (58) and (59) since both $k\,x^{\lambda'}\,(x-1)^{\nu'}$ and $k\,x^{\lambda''}\,(x-1)^{\nu''}$ are members of the family. A necessary result is

$$s = -\frac{\lambda' - \lambda''}{\mu' - \mu''}. \tag{60}$$

But if (60) is satisfied, our family must be of Class 3; for the only solutions of (55) in the doubtful cases are, as we have noted, either (58) or (59); but (60) combined with either gives the other, so that if (60) is satisfied, our family must in fact have two members $k\,x^{\lambda'}\,(x-1)^{\nu'}$ and $k\,x^{\lambda''}\,(x-1)^{\nu''}$, and must therefore be of Class 3.

From the foregoing we easily see that the following relations are necessary and sufficient, for each class, that a family under the doubtful cases belong to the class designated:

$$\begin{aligned}
&\text{Class 1.}\quad A = -\lambda' - \mu' s, \quad s \neq -\frac{\lambda' - \lambda''}{\mu' - \mu''}.\\
&\quad\text{“}\quad 2.\quad A = -\lambda'' - \mu'' s, \quad s \neq -\frac{\lambda' - \lambda''}{\mu' - \mu''}.\\
&\quad\text{“}\quad 3.\quad s = -\frac{\lambda' - \lambda''}{\mu' - \mu''}.
\end{aligned}$$

Type III. The doubtful cases here belong under two headings characterized by the relations $\lambda' + \mu' + \nu' = 0$, and $\lambda' + \mu' + \nu' = \lambda' - \lambda''$, respectively.

When $\lambda' + \mu' + \nu' = 0$, the two roots of (55) are $-\lambda' - \mu' s$ and $-\lambda'' - \mu'' s$. As noted in the discussion for Type II, the relation $A = -\lambda' - \mu' s$ is a necessary and sufficient condition that $k\,x^{\lambda'}\,(x-1)^{\nu'}$ be a member of the family, and $A = -\lambda'' - \mu'' s$ is a necessary and sufficient condition that $k\,x^{\lambda''}\,(x-1)^{\nu''}$ be a member of the family. These two are, then, the only possible everywhere fundamental branches; if a family is of Class 1, it has the former branch; if of Class 2, the latter; if of Class 3, the latter. The necessary and sufficient conditions for each class are:

$$\begin{aligned}
&\text{Class 1.}\quad A = -\lambda' - \mu' s, \quad s \neq -\frac{\lambda' - \lambda''}{\mu' - \mu''}.\\
&\quad\text{“}\quad 2.\quad A = -\lambda'' - \mu'' s, \quad \lambda' = \lambda''.\\
&\quad\text{“}\quad 3.\quad A = -\lambda'' - \mu'' s, \quad \lambda' \neq \lambda''.
\end{aligned}$$

Since $s \neq 0, \infty, 1$, we omit for Class 2 the condition $s \neq -\dfrac{\lambda' - \lambda''}{\mu' - \mu''}$. Similarly hereafter.

If $\lambda' + \mu' + \nu' = \lambda' - \lambda''$, *i.e.*, $\lambda'' + \mu' + \nu' = 0$, the two roots of (55) are $-\lambda'' - \mu' s$ and $-\lambda' - \mu'' s$. For a given family here, $A = -\lambda'' - \mu' s$ is necessary and sufficient that $k\, x^{\lambda''}(x-1)^{\nu'}$ be a member, while $A = -\lambda'' - \mu' s$ is necessary and sufficient that $k\, x^{\lambda'}(x-1)^{\nu''}$ be a member. Of these two, which are the only possible everywhere fundamental branches, families of Class 1 have the former, those of Class 2 the latter, and those of Class 3 the former. We may, then, characterize the classes as follows:

$$\begin{array}{lll} \text{Class 1.} & A = -\lambda'' - \mu' s, & \lambda' = \lambda''. \\ \text{\quad `` \ 2.} & A = -\lambda' - \mu'' s, & s \neq \dfrac{\lambda' - \lambda''}{\mu' - \mu''}. \\ \text{\quad `` \ 3.} & A = -\lambda'' - \mu' s, & \lambda' \neq \lambda''. \end{array}$$

Type IV. The doubtful cases come under four heads, namely: (a) $\lambda' + \mu' + \nu' = 0$, ($b$) $\lambda' + \mu' + \nu' = \lambda' - \lambda'' \neq 0$, ($c$) $\lambda' + \mu' + \nu' = \mu' - \mu'' \neq 0$, ($d$) $\lambda' + \mu' + \nu' = \nu' - \nu'' \neq 0$. These we take up in order.

(a) $\lambda' + \mu' + \nu' = 0$. Here $\lambda'' + \mu'' + \nu'' = 0$, and since $\lambda' - \lambda''$, $\mu' - \mu''$, $\nu' - \nu''$ are positive integers or zero, they must in this case all be zero. The three regular singular points are each logarithmic, so that we have to do here with only the classes other than 1, 2, 3, 4; and only when $\lambda' + \mu' + \nu' = 0$ is such a class possible. We have fundamental branches

$$x^{\lambda'}(x-1)^{\nu'},$$
$$x^{\lambda'}(x-1)^{\nu'}\,[\kappa'' \log x + \kappa' \log (x-1)],$$

(see page 46), where $\kappa' \neq 0$, $\kappa'' \neq 0$, and $\kappa' + \kappa'' \neq 0$. Each class here is characterized by the ratio $\frac{\kappa''}{\kappa'}$. This ratio we can find in terms of s by substituting branches 1 and $\kappa'' \log x + \kappa' \log (x-1)$ in the functional determinant of the family $x^{-\lambda'}(x-1)^{-\nu'}\, Q$, thus obtaining the identity

$$\begin{vmatrix} 1 & 0 \\ \kappa'' \log x + \kappa' \log (x-1) & \dfrac{\kappa''}{x} + \dfrac{\kappa'}{x-1} \end{vmatrix} \equiv k\, x^{\lambda' + \lambda'' - 2\lambda' - 1}\,(x-1)^{\nu' + \nu'' - 2\nu' - 1}\,(x - s) \equiv k\,\frac{x - s}{x\,(x-1)}.$$

Hence we have

$$\frac{\kappa''}{\kappa'} = -\frac{s}{s-1}.$$

Each value of s (*other than* 0, 1, ∞) *gives a class.* For all these classes the accessory parameter A has the value $-\lambda' - \mu' s$. We see here that *classes actually exist corresponding to every value of* $\frac{\kappa''}{\kappa'}$.

(b) If $\lambda' + \mu' + \nu' = \lambda' - \lambda'' \neq 0$, *i. e.*, $\lambda'' + \mu' + \nu' = 0$, $\lambda' + \mu' + \nu' \neq 0$, the two roots of (55) are $-\lambda'' - \mu' s$ and $-\lambda' - \mu'' s$. We hardly need give here the details of a discussion which will be so similar to that given for Type III. The criteria are:

Class 1. $A = -\lambda'' - \mu' s, \quad s \neq \dfrac{\lambda' - \lambda''}{\mu' - \mu''}.$

“ 2. $A = -\lambda' - \mu'' s, \quad \mu' \neq \mu'', \quad \nu' = \nu''.$

“ 3. $A = -\lambda' - \mu'' s, \quad \mu' = \mu'', \quad \nu' \neq \nu''.$

“ 4. $A = -\lambda' - \mu'' s, \quad \mu' \neq \mu'', \quad \nu' \neq \nu''.$

No other classes possible.

(c) $\lambda' + \mu' + \nu' = \mu' - \mu'' \neq 0$. The two roots of (55) are $-\lambda'' - \mu' s$ and $-\lambda' - \mu'' s$. The criteria are:

Class 1. $A = -\lambda'' - \mu' s, \quad \lambda' \neq \lambda'', \nu' = \nu''.$

“ 2. $A = -\lambda' - \mu'' s, \quad s \neq \dfrac{\lambda' - \lambda''}{\mu' - \mu''}.$

“ 3. $A = -\lambda'' - \mu' s, \quad \lambda' = \lambda'', \quad \nu' \neq \nu''.$

“ 4. $A = -\lambda'' - \mu' s, \quad \lambda' \neq \lambda'', \quad \nu' \neq \nu''.$

(d) $\lambda' + \mu' + \nu' = \nu' - \nu'' \neq 0$. The two roots of (55) are $-\lambda' - \mu' s$ and $-\lambda'' - \mu'' s$. The criteria are:

Class 1. $A = -\lambda'' - \mu'' s, \quad \lambda' \neq \lambda'', \quad \mu' = \mu''.$

“ 2. $A = -\lambda'' - \mu'' s, \quad \lambda' = \lambda'', \quad \mu' \neq \mu''.$

“ 3. $A = -\lambda' - \mu' s, \quad s \neq -\dfrac{\lambda' - \lambda''}{\mu' - \mu''}.$

“ 4. $A = -\lambda'' - \mu'' s, \quad \lambda' \neq \lambda'', \quad \mu' \neq \mu''.$

§ 3. EXPRESSION OF A Q FUNCTION OF ORDER 1 AS A SUM OF TWO KINDRED P FUNCTIONS.

The exponent scheme of a Q family of order 1 being

$$Q\left(\begin{array}{cccc} 0 & \infty & 1 & \\ \lambda' & \mu' & \nu' & \\ \lambda'' & \mu'' & \nu'' & x \end{array}\right), \tag{61}$$

Ritter states that a function belonging to this family is in general expressible linearly in terms of two kindred P functions chosen from one of the following triples of P families: In the first triple, the three families have the same notation scheme for the exponents as Q, except that the first family has $\lambda' + 1$ in place of λ', the second $\mu' + 1$ in place of μ', and the third $\nu' + 1$ in place of ν'; in the second triple, there are similar changes in the second row of exponents in (61). This representation he does not prove to hold for all cases, but merely gives an example for a family all of whose points are ordinary. Our formulæ of Part II, C, show that this expression is valid, provided (1) that we can choose two such P families kindred to Q, and (2) that the two P families chosen, and Q, have a common connecting formula (including the associated constants C) *in which no two members of any basis have the same exponent* (except in the case of equal exponents). The reason for this last condition is obvious from the remarks of page 48.

The first condition is violated by Q functions of Type II, Class 3, and of the classes of Type IV other than 1, 2, 3, 4, since P families cannot belong to those classes. Both conditions are always satisfied for Type I; and an inspection of each class for which P families are possible under Types II, III, IV will show that the first condition can always be satisfied for a Q family of any of those classes. Formula (50) shows that the degrees of the coefficients in (51) taken for functions of the Q family and two kindred P families chosen as above indicated, must be zero, provided the second condition above is satisfied. The reader may assure himself by an examination of each class that this second condition is also satisfied except in the *doubtful cases* noted in the preceding section; and that here the only violation is in Class 3 of Type III, and Class 4 of Type IV, when the two values of A coincide, the family being then completely determined by its exponent scheme and apparently singular point. Here the formulæ of Part II, C, fail to give the desired representation, which nevertheless exists. In fact if P' and P'' be two functions chosen as above indicated, it is easy to show that constants α and β always exist such that the function

$$Q = \alpha P' + \beta P'',$$

which evidently has the desired exponent scheme, has also the characteristic apparently singular point.

A Q *function of order* 1 *of any class possible for* P *families can always be expressed linearly in terms of kindred* P *functions.*

We have thus illustrated some of the difficulties that arise in a first generalization from P families. Our methods are sufficient for the discussion of further cases, but conditions become so complicated that a similar treatment of Q families of order 2, or of higher order, will not be attempted here.